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A Third Monte Carlo Sampler

(A Revision and Extension of Samplers I and II)

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CONTENTS

	PAGE
ABSTRACT	1
FOREWORD	1
FORMULAS	2
A Note on Statistics	17
D-INDEX	19
Discrete Densities	19
C-INDEX	45
Continuous Densities	45
R-INDEX	123
Rejection Techniques	123
ACKNOWLEDGMENTS	147
REFERENCES	147
APPENDIX	149

A THIRD MONTE CARLO SAMPLER
(A Revision and Extension of Samplers I and II)

by

C. J. Everett and E. D. Cashwell

ABSTRACT

Methods are given for sampling some standard probability densities by means of machine generated "random numbers." The probability theory underlying each device is briefly indicated. The present collection embodies the densities of the first two Samplers, and an attempt has been made to render the explanations less terse and more understandable. Some additional methods and new densities have been included. No attempt has been made to quote all original sources, and no claim to priority is intended in any case, our sole object being to provide a handbook of sampling devices.

FOREWORD

In all cases, the density to be sampled is followed by one or more rules (R_x) for choice of the variable, in terms of random numbers $r_0, r_1, \ldots$ uniformly distributed on the interval (0,1). A justification (J) for the rule is given, frequently supported by various formulas (F). Notes supply additional details, and often refer to the relation with other densities. The indices (D,C,R) for discrete (D) and continuous (C) densities, and for various rejection techniques (R), provide "key words" which may help in locating a desired density, but details in this direction are omitted. Numbers in square brackets [ ] refer to the references at the end of the report.

FORMULAS

F1. $$\frac{d}{du}\int_{g(u)}^{h(u)} f(u,\nu)\,d\nu = h'(u)f(u,h(u)) - g'(u)f(u,g(u)) + \int_{g(u)}^{h(u)} \frac{\partial}{\partial u} f(u,\nu)\,d\nu.$$

For an idea of the proof, note that

$$\int_{g(u+\Delta u)}^{h(u+\Delta u)} f(u+\Delta u,\nu)\,d\nu - \int_{g(u)}^{h(u)} f(u,\nu)\,d\nu = \int_{g(u+\Delta u)}^{g(u)} + \int_{g(u)}^{h(u)}$$

$$+ \int_{h(u)}^{h(u+\Delta u)} f(u+\Delta u,\nu)\,d\nu - \int_{g(u)}^{h(u)} f(u,\nu)\,d\nu = \int_{h(u)}^{h(u+\Delta u)}$$

$$- \int_{g(u)}^{g(u+\Delta u)} f(u+\Delta u,\nu)\,d\nu + \int_{g(u)}^{h(u)} [f(u+\Delta u,\nu) - f(u,\nu)]\,d\nu$$

$$= \Delta h f(u+\Delta u, h(\overline{u})) - \Delta g f(u+\Delta u, g(\overline{\overline{u}})) + \int_{g(u)}^{h(u)} [f(u+\Delta u,\nu)$$

$$- f(u,\nu)]\,d\nu.$$

One divides by Δu, and takes the limit as $\Delta u \to 0$.

F2.A. For densities p_1, p_2 on $(0,\infty)$, and $0 < u < \infty$,

$$D_1 \equiv \frac{d}{du} \int_{\left\{\begin{matrix}\nu_1+\nu_2\le u\\ \nu_i>0\end{matrix}\right\}} p_1(\nu_1)p_2(\nu_2)\,d\nu_1\,d\nu_2$$

$$= \frac{d}{du}\int_0^u \left\{ p_1(\nu_1) \int_0^{u-\nu_1} p_2(\nu_2)\,d\nu_2 \right\} d\nu_1$$

$$\equiv \frac{d}{du}\int_0^u \{f(u,\nu_1)\}\, d\nu_1 = 1 \cdot f(u,u) - 0 \cdot f(u,0)$$

$$+ \int_0^u \frac{\partial}{\partial u}\{f(u,\nu_1)\}\, d\nu_1 = 0 - 0$$

$$+ \int_0^u \left\{ p(\nu_1)\, \frac{\partial}{\partial u}\int_0^{u-\nu_1} p_2(\nu_2)\, d\nu_2 \right\} d\nu_1 \text{ by F1.}$$

With ν_1 constant,

$$\frac{d}{du}\int_0^{u-\nu_1} p_2(\nu_2)\, d\nu_2 = 1 \cdot p_2(u-\nu_1) - 0 \cdot p_2(0)$$

$$+ \int_0^{u-\nu_1} \frac{\partial}{\partial u} p_2(\nu_2)\, d\nu_2 = p_2(u - \nu_1) - 0 + 0 = p_2(u - \nu_1).$$

Hence,

$$D_1 = \int_0^u p_1(\nu_1)p_2(u - \nu_1)\, d\nu_1 \quad .$$

B. For even densities $p_1(\nu_1)$, $p_2(\nu_2)$ on $(-\infty,\infty)$, and $-\infty < u < \infty$, one has

$$D_2 \equiv \frac{d}{du}\int_{\{\nu_2/\nu_1 \leq u\}} p_1(\nu_1)p_2(\nu_2)\, d\nu_1\, d\nu_2$$

$$= \frac{d}{du}\left\{ \int_{-\infty}^0 d\nu_1\, p_1(\nu_1) \int_{u\nu_1}^{\infty} p_2(\nu_2)\, d\nu_2 + \int_0^{\infty} d\nu_1\, p_1(\nu_1) \int_{-\infty}^{u\nu_1} p_2(\nu_2)\, d\nu_2 \right\}$$

$$= \int_{-\infty}^{0} d\nu_1 p_1(\nu_1) \cdot (-\nu_1) \cdot p_2(u\nu_1) + \int_{0}^{\infty} d\nu_1\ p_1(\nu_1) \cdot (\nu_1) \cdot p_2(u\nu_1)$$

$$= \int_{0}^{\infty} d\nu_1'\ p_1(-\nu_1') \cdot (\nu_1')p_2(-u\nu_1') + \int_{0}^{\infty} d\nu_1\ p_1(\nu_1)(\nu_1)p_2(u\nu_1)$$

$$= \int_{0}^{\infty} d\nu_1'\ p_1(\nu_1')(\nu_1')p_2(u\nu_1') + \int_{0}^{\infty} d\nu_1\ p_1(\nu_1)(\nu_1)p_2(u\nu_1)$$

$$= 2\int_{0}^{\infty} d\nu_1\ p_1(\nu_1)(\nu_1)p_2(u\nu_1) \quad .$$

F3. $p(\nu) = e^{-\xi}\xi^{\nu}/\nu!$; $\nu = 0,1,2,\ \ldots,\ \xi > 0$, is the Poisson density, with distribution

$$P(n) = \sum_{0}^{n} p(\nu) = e^{-\xi} \sum_{0}^{n} \xi^{\nu}/\nu! \quad .$$

A. $$\frac{1}{(n-1)!}\int_{\xi}^{\infty} x^{n-1}e^{-x}\,dx = e^{\xi}\sum_{0}^{n-1} \xi^{\nu}/\nu! = P(n-1),\ n = 1,2,\ \ldots \quad .$$

(Induction on n, integration by parts.) Thus the Poisson distribution is an incomplete Γ-function (see F4.).

B. For $\xi,\eta > 0$, one has

$$\int_{\eta}^{\infty} y^{n-1}e^{-(\xi/\eta)y}\,dy = (\eta/\xi)^n \int_{\xi}^{\infty} x^{n-1}e^{-x}\,dx \qquad (y = \eta x/\xi.)$$

$$= (\eta/\xi)^n(n-1)!\ e^{-\xi}\sum_{0}^{n-1} \xi^{\nu}/\nu. \qquad \text{(By A.)}$$

C. $D_\xi \equiv \int_1^\infty y^{n-1} e^{-\xi y}\, dy = \xi^{-n}(n-1)!\, e^{-\xi} \sum_0^{n-1} \xi^\nu/\nu! = \xi^{-n}(n-1)!\, e^{-\xi} S_\xi$,

where $S_\xi \equiv \sum_0^{n-1} \xi^\nu/\nu!$ (η = 1 in B.)

F4. $\Gamma(n) \equiv \int_0^\infty u^{n-1} e^{-u}\, du$; n real > 0, is the Γ-function.

A. $\Gamma(n) = 2\int_0^\infty \nu^{2n-1} e^{-\nu^2}\, d\nu.$ $(u = \nu^2)$

B. $\Gamma(1/2) = \pi^{1/2}.$ (See F5.G.)

$\Gamma(m)\Gamma(1-m) = \pi/\sin m\pi$, $0 < m < 1$. (Not easy. See [23; p. 89].)

C. $\Gamma(n+1) = n\Gamma(n)$. (Integration by parts.)

D. For integral n = 0,1,2, ..., $\Gamma(n+1) = n!$, $0! \equiv 1$.

E. $2^{2m-1}\Gamma(m)\Gamma(m+1/2) = \Gamma(1/2)\Gamma(2m)$; m real > 0.

(Legendre's identity. See F5H.)

F5. $B(m,n) \equiv \int_0^1 v^{m-1}(1-v)^{n-1}\, dv$; m,n real > 0, is the B-function.

A. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$ $(v = \sin^2\theta.)$

$B(1/2,1/2) = \pi.$

B. $B(m,n) = \int_0^\infty z^{m-1}\, dz/(1+z)^{m+n}$ $(z = v/(1-v).)$

C. $$\int_0^1 x^{m-1}(1 - x)^{n-1}\, dx/(x + a)^{m+n} = B(m,n)/(1 + a)^m a^n;$$

$a,m,n > 0$. (Let $x = y/(1 + y)$ and $y = az/(1 + a)$.)

D. $$\int_{-a}^{a} (a + x)^{m-1}(a - x)^{n-1}\, dx = (2a)^{m+n-1}B(m,n); \; a,m,n > 0.$$

($x = a/(2v - 1)$.)

E. $$B(m,n) = \int_0^1 (y^{m-1} + y^{n-1})\, dy/(1 + y)^{m+n}.$$

(From B, $$B(m,n) = \int_0^1 z^{m-1}\, dz/(1 + z)^{m+n} + \int_1^\infty z^{m-1}\, dz/(1 + z)^{m+n}$$

$$= \int_0^1 y^{m-1}\, dy/(1 + y)^{m+n} + \int_0^1 y^{n-1}\, dy/(1 + y)^{m+n},$$

for $z = 1/y$.)

F. $B(m,n) = \Gamma(m)\Gamma(n)/\Gamma(m + n), B(m,n) = B(n,m)$. (By F4.A.,

$$\Gamma(m)\Gamma(n) = 2\int_0^\infty y^{2m-1}e^{-y^2}\, dy \cdot 2\int_0^\infty x^{2n-1}e^{-x^2}\, dx$$

$$= 2\int_0^\infty \rho^{2(m + n)-1}e^{-\rho^2}\, d\rho \cdot 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= \Gamma(m + n)B(m,n).)$$

G. $\Gamma(1/2) = \pi^{1/2}$. ($m = n = 1/2$ in F.)

H. Proof of Legendre's identity F4.E.:

$$B(m,m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta = (1/2)^{2m-1}\int_0^{\pi/2} \sin^{2m-1} 2\theta \, d(2\theta)$$

$$= (1/2)^{2m-1}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi = (1/2)^{2m-1} \cdot 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$$

$$= (1/2)^{2m-1}B(m,1/2).$$

Hence, $\Gamma(m)\Gamma(m)/\Gamma(2m) = (1/2)^{2m-1}\Gamma(m)\Gamma(1/2)/\Gamma(m + 1/2)$.

F6. $$\int_{\left\{\sum_1^n \mu_i \le 1,\ \mu_i > 0\right\}} \prod_1^n \mu_i^{s_i - 1} \, d\mu_i = \prod_1^n \Gamma(s_i)/\Gamma\left(1 + \sum_1^n s_i\right); \quad s_i > 0 .$$

Proof by induction on n. For n = 1, see F4.C. Letting $\mu = \mu_{n+1}$,

$s = s_{n+1}$, $S = \sum_1^n s_i$, one has

$$\int_{\left\{\sum_1^{n+1} \mu_i \le 1,\ \mu_i > 0\right\}} \prod_1^{n+1} \mu_i^{s_i - 1} \, d\mu_i = \int_0^1 \mu^{s-1} \, d\mu \int_{\left\{\sum_1^n \mu_i/(1-\mu) \le 1\right\}} \prod_1^n \mu_i^{s_i - 1} \, d\mu_i$$

$$= \int_0^1 \mu^{s-1}\, d\mu\, (1 - \mu)^S \int_{\left\{\sum_1^n \nu_i \le 1\right\}} \prod_1^n \nu_i^{s_i - 1}\, d\nu_i \qquad (\nu_i = \mu_i/(1 - \mu))$$

$$= B(s, S + 1) \prod_1^n \Gamma(s_i)/\Gamma(1 + S) = \frac{\Gamma(s)\Gamma(S + 1)}{\Gamma(s + S + 1)} \frac{\prod_1^n \Gamma(s_i)}{\Gamma(1 + S)}$$

$$= \frac{\Gamma(s) \prod_1^n \Gamma(s_i)}{\Gamma(s + S + 1)}$$

where we have used F5.F. for $B(s, S + 1)$.

F7. $$V(u) \equiv \int_{\left\{\sum_1^n \nu_i \le u,\ \nu_i > 0\right\}} \prod_1^n d\nu_i = u^n/n!$$

Let $\nu_i = u\mu_i$ and use F6.) Hence $A(u) \equiv dV/du = u^{n-1}/(n - 1)!$

F8. $$V(u) \equiv \int_{\left\{\left(\sum_1^N \nu_i^2\right)^{1/2} \le u,\ \nu_i > 0\right\}} \prod_1^N d\nu_i = \pi^{N/2} u^N / 2^{N-1} N \Gamma(N/2).$$

(Let $\nu_i = u\mu_i^{1/2}$ and use F6.). Hence $A(u) \equiv dV/du = \pi^{N/2}u^{N-1}/2^{N-1}\Gamma(N/2)$.
Note. $V(u) = 2\pi^{N/2}u^N/N\Gamma(N/2)$ is the volume of the full N-sphere of radius u, $A(u) = 2\pi^{N/2}u^{N-1}/\Gamma(N/2)$ its area. For the full unit sphere, $V(1) = 2\pi^{N/2}/N\Gamma(N/2)$, and $A(1) = 2\pi^{N/2}/\Gamma(N/2) = \int_\Omega d\Omega$, where $\Omega = (\omega_1, \ldots, \omega_N)$ is the direction in N-space.

F9. $\zeta(n) \equiv \sum_1^\infty 1/j^n$; n real > 1, is the ζ-function.

A. $\zeta_a(n) \equiv \sum_1^\infty (-1)^{j+1}/j^n = (1 - (1/2^{n-1}))\zeta(n)$,

$\zeta_u(n) \equiv \sum_1^\infty 1/(2j - 1)^n = (1 - (1/2^n))\zeta(n)$.

Proof. By subtraction and addition of the series

$\zeta(n) = 1 + 1/2^n + 1/3^n + 1/4^n + \ldots$

$\zeta_a(n) = 1 - 1/2^n + 1/3^n - 1/4^n + \ldots$

one obtains $\zeta(n) - \zeta_a(n) = (2/2^n)\zeta(n)$,

and

$\zeta(n) + \zeta_a(n) = 2\zeta_u(n)$, whence the result.

B. It is known that $\zeta(2n) = (-1)^{n-1}(2\pi)^{2n}B_{2n}/2(2n)!$; $n = 1,2,3, \ldots$, where $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, ... are the Bernoulli numbers. Thus $\zeta(2) = \pi^2/6$, $\zeta(4) = \pi^4/90$, $\zeta(6) = \pi^6/945$, ... Computation shows that $\zeta(3) = 1.2021 \ldots$, and

C. $\zeta_u(2) = (1 - (1/2^2))\zeta(2) = (3/4)(\pi^2/6) = \pi^2/8.$

D. For $n > 1$,

$$\int_0^\infty \nu^{n-1}\, d\nu/(e^\nu - 1) = \int_0^\infty \nu^{n-1}\, d\nu\, e^{-\nu}/(1 - e^{-\nu}) = \int_0^\infty \nu^{n-1}\, d\nu \sum_1^\infty e^{-j\nu}$$

$$= \sum_1^\infty (1/j^n) \int_0^\infty (j\nu)^{n-1} e^{-j\nu}\, d(j\nu) = \zeta(n)\Gamma(n) \quad .$$

E. For $n > 1$, $\int_0^\infty 2\mu^{2n-1}\, d\mu/(e^{\mu^2} - 1) = \zeta(n)\Gamma(n)$. Let $\mu = \nu^{1/2}$.

F10. Define $F_i = (a_1 \ldots a_n)/(a_1 - a_i) \ldots (a_{i-1} - a_i)(a_{i+1} - a_i) \ldots (a_n - a_i)$ for $n \geq 2$, and distinct $a_i > 0$.

A. $\sum_1^n F_i = 0.$

Proof. For $f(z) = (a_1 \ldots a_n)/(a_1 - z) \ldots (a_n - z)$, one has

$(1/2\pi i) \int_c f(z)\, dz = \sum_1^n \mathrm{Res}(a_i)$, where c is any circle of radius

$R > \max a_i$, and the residue $\mathrm{Res}\,(a_i) = \lim_{z \to a_i} (z - a_i) f(z) = -F_i$.

Hence $(1/2\pi i) \int_c f(z)\, dz = -\sum_1^n F_i$. But

$$\left|\int_c f(z)\, dz\right| \leq 2\pi R \max_c |f(z)| \to 0 \text{ as } R \to \infty, \text{ since } n \geq 2.$$

B. $\sum_{1}^{n} F_i/a_i = 1.$

Proof. The function $g(z) = (a_1 \ldots a_n)/z(a_1 - z) \ldots (a_n - z)$ has residues at $z = 0, a_1, \ldots, a_n$ where $\mathrm{Res}(0) = \lim_{z\to 0} zg(z) = 1$, and $\mathrm{Res}(a_i) = \lim_{z\to a_i} (z - a_i)g(z) = - F_i/a_i$. Hence by the type of argument in

A, the sum of all residues is $1 - \sum_{1}^{n} F_i/a_i = 0.$

F11. For $0 < \Lambda \leq 1$, $n > 1$,

$$\int_0^\infty \nu^{n-1}\, d\nu/(\Lambda^{-1}e^{\nu} + 1) = \int_0^\infty \nu^{n-1}\, d\nu \Lambda e^{-\nu}/(1 + \Lambda e^{-\nu})$$

$$= \sum_{1}^{\infty} (-1)^{j+1}(\Lambda^j/j^n) \int_0^\infty j^n \nu^{n-1} e^{-j\nu}\, d\nu$$

$$\equiv \zeta_a(\Lambda,n)\Gamma(n), \text{ where } \zeta_a(\Lambda,n) \equiv \sum_{1}^{\infty} (-1)^{j+1}\Lambda^j/j^n \quad .$$

F12. For $0 < \Lambda \leq 1$, $n > 1$,

$$\int_0^\infty \nu^{n-1}\, d\nu\, \Lambda e^{-\nu}/(1 - \Lambda^2 e^{-2\nu}) = \sum_{1}^{\infty} (\Lambda^{2j-1}/(2j - 1)^n)$$

$$\cdot \int_0^\infty (2j - 1)^n \nu^{n-1} e^{-(2j-1)\nu}\, d\nu = \zeta_u(\Lambda,n)\Gamma(n), \text{ where } \zeta_u(\Lambda,n)$$

$$\equiv \sum_{1}^{\infty} \Lambda^{2j-1} /(2j - 1)^n \; .$$

F13. Define $K_N(u) \equiv \int_0^\infty \cosh N\theta e^{-u \cosh\theta} \, d\theta$; $(0,\infty)$, $N \geq 0$.

A. $$K_N(u) = 1/2 \int_{-\infty}^{\infty} e^{-N\theta} e^{-u \cosh\theta} \, d\theta. \quad \text{(From definition.)}$$

B. $$K_N(u) = 2^{-(N+1)} u^N \int_0^\infty x^{-(N+1)} e^{-(x + (u^2/4x))} \, dx. \quad (e^\theta = 2x/u \text{ in A.})$$

C. $$K_N(u) = \frac{\Gamma(1/2)u^N}{2^N \Gamma(N + 1/2)} \int_1^\infty (\nu^2 - 1)^{N-1/2} e^{-u\nu} \, d\nu \qquad \text{(Cf.[12; p. 185])}$$

$$= \frac{\Gamma(1/2)u^N}{2^N \Gamma(N + 1/2)} \int_0^1 x^{-(2N+1)} (1 - x^2)^{N-1/2} e^{-u/x} \, dx. \qquad (\nu = 1/x.)$$

D. $$K_N(2v^{1/2}) = 2^{-1} v^{N/2} \int_0^\infty x^{-(N+1)} e^{-(x + (v/x))} \, dx. \qquad (u = 2v^{1/2} \text{ in B.})$$

F14. For $n, \xi > 0$,

$$\int_0^\infty x^{n-1} e^{-\xi(x^2+1)^{1/2}} \, dx = \int_1^\infty \nu \, (\nu^2 - 1)^{(n/2)-1} e^{-\xi\nu} \, d\nu \qquad ((x^2 + 1)^{1/2} = \nu)$$

$$= (\xi/n) \int_1^\infty (\nu^2 - 1)^{n/2} e^{-\xi\nu} \, d\nu$$

(parts: $u = e^{-\xi\nu}$, $dv = \nu(\nu^2 - 1)^{(n/2)-1} \, d\nu$)

$$= \frac{\Gamma(n/2)}{\Gamma(1/2)} \left(\frac{2}{\xi}\right)^{\frac{n-1}{2}} K_{\frac{n+1}{2}}(\xi).$$

(F13.C., 1st eqn., with $u = \xi$, $N = (n + 1)/2$.)

F15. For n, $a > 0$, $0 < \Lambda \leq 1$,

$$\int_0^\infty x^{n-1}\, dx \Big/ \left(\Lambda^{-1} e^{a(x^2+1)^{1/2}} + 1\right)$$

$$= \int_0^\infty x^{n-1} \Lambda e^{-a(x^2+1)^{1/2}}\, dx \Big/ \left(1 + \Lambda e^{-a(x^2+1)^{1/2}}\right)$$

$$= \sum_1^\infty (-1)^{j+1} \Lambda^j \int_0^\infty x^{n-1} e^{-ja(x^2+1)^{1/2}}\, dx$$

$$= 2^{\frac{n-1}{2}} (\Gamma(n/2)/\Gamma(1/2)) \sum_1^\infty (-1)^{j+1} \Lambda^j K_{\frac{n+1}{2}}(ja)/(ja)^{\frac{n-1}{2}}. \quad \text{(See F14.)}$$

F16. For n, $a > 0$, $0 < \Lambda \leq 1$,

$$\int_1^\infty \nu^{n-1} \Lambda e^{-a\nu}\, d\nu/(1 - \Lambda^2 e^{-2a\nu}) = \sum_1^\infty \Lambda^{2j-1} \int_1^\infty \nu^{n-1} e^{-(2j-1)a\nu}\, d\nu$$

$$= \sum_1^\infty \Lambda^{2j-1} D_{(2j-1)a},$$

where D_ξ is defined and evaluated in F3.C.

F17. For $K_N(u)$ as in F13., and $0 \leq N < n$,

$$\int_0^\infty u^{n-1}K_N(u)\,du = (\Gamma(1/2)/2^N\Gamma(N + 1/2)) \int_1^\infty d\nu\ (\nu^2 - 1)^{N-1/2}$$

$$\cdot \int_0^\infty u^{n+N-1}e^{-u\nu}\,du = (\Gamma(1/2)\Gamma(n + N)/2^N\Gamma(N + 1/2)) \qquad \text{(F13.C.)}$$

$$\cdot \int_1^\infty \nu^{-(n+N)}(\nu^2 - 1)^{N-1/2}\,d\nu = (\Gamma(1/2)\Gamma(n + N)/2^{N+1}\Gamma(N + 1/2)) \quad \text{(F4.)}$$

$$\cdot \int_0^1 \xi^{((n-N)/2)-1}(1 - \xi)^{(N+1/2)-1}\,d\xi \qquad (\nu = \xi^{-1/2})$$

$$= (\Gamma(1/2)\Gamma(n + N)/2^{N+1}$$

$$\cdot\ \Gamma(N + 1/2))B((n - N)/2,\ N + 1/2) = \Gamma(1/2)\Gamma(n + N) \qquad \text{(F5.)}$$

$$\cdot\ \Gamma((n - N)/2)/2^{N+1}\Gamma((n + N + 1)/2) = 2^{n-2}\Gamma((n - N)/2)\Gamma((N + n)/2).$$

(F4.E. with $m = (n + N)/2$.)

F18. Define $E_N(u) = \int_1^\infty \nu^{-N}e^{-u\nu}\,d\nu$; $(0,\infty)$, $N \geq 0$. (Cf. [1].)

A. $$E_N(u) = \int_0^1 x^{N-2}e^{-u/x}\,dx. \qquad (\nu = 1/x.)$$

F19. For $E_N(u)$ as in F18., and $N \geq 0$, $n > 0$, $n + N > 1$,

$$\int_0^\infty u^{n-1}E_N(u)\,du = \int_1^\infty d\nu\ \nu^{-N} \int_0^\infty u^{n-1}e^{-u\nu}\,du \qquad \text{(F18.)}$$

$$= \Gamma(n) \int_1^\infty d\nu\ \nu^{-(n + N)} = \Gamma(n)/(n + N - 1). \qquad \text{(F4.)}$$

F20. $\#(S_1 \cup \ldots \cup S_k) = \sum_{\binom{k}{1}} \#S_{i_1} - \sum_{\binom{k}{2}} \#\left(S_{i_1} S_{i_2}\right) + \ldots + (-1)^{k-1} \#(S_1 \ldots S_k).$

This is the "inclusion-exclusion" principle, which gives the number (#) of elements in the union $S_1 \cup \ldots \cup S_k$ of k subsets of a set in terms of the number of elements in the intersections of these sets, taken any number at a time, there being $\binom{k}{j}$ terms in the j-th sum. The result is easily proved by induction on $k \geq 2$, with the obvious basis $\#(S_1 \cup S_2) = (\#S_1 + \#S_2) - \#(S_1 S_2)$.

F21. $\sum_{q=1}^{\infty} \phi(q) y^q/(1 - y^q) \equiv y/(1 - y)^2; \; |y| < 1.$

In this identity of Liouville, $\phi(q)$ is Euler's ϕ-function, which counts the number of integers in the set $\{1, 2, \ldots, q\}$ which are prime to q. Clearly $\phi(1) = 1$, and if $q = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ is the standard factorization of $q \geq 2$ into primes, then one knows that $\phi(q) = q(1 - 1/p_1) \ldots (1 - 1/p_k)$. It can be proved that

$$\sum_{d|q} \phi(d) = q,$$

where d ranges over all positive divisors of q. The latter property of $\phi(q)$ permits an easy proof of Liouville's identity. The right side is

$$y/(1 - y)^2 = y(1 + y + y^2 + \ldots)^2 = y + 2y^2 + 3y^3 + 4y^4 + \ldots .$$

On the left we have

$$\phi(1)\{y^1) + (y^1)^2 + (y^1)^3 + (y^1)^4 + \ldots\}$$
$$+ \phi(2)\{(y^2) + (y^2)^2 + (y^2)^3 + (y^2)^4 + \ldots\}$$
$$+ \phi(3)\{(y^3) + (y^3)^2 + (y^3)^3 + (y^3)^4 + \ldots\} + \ldots .$$

A particular power y^q of y occurs in just those terms $\phi(d)\{(y^d) + (y^d)^2 + (y^d)^3 + \ldots\}$ for which d divides q. Hence on the left, the coefficient of y^q is

$$\sum_{d|q} \phi(d) = q,$$

and the identity follows.

F22. $\int_0^\infty \sin x \, dx/x = \pi/2$. (Not easy. Cf. [20; p. 88].) From this,

$$\int_0^\infty \sin^2 x \, dx/x^2 = -(\sin x/x) \sin x \Big]_0^\infty + \int_0^\infty \sin 2x \, dx/x$$

$$= 0 + \int_0^\infty \sin y \, dy/y = \pi/2,$$

where we have integrated by parts, with $u = \sin^2 x$, $dv = dx/x^2$.

F23. $$\int_{-\infty}^\infty x \operatorname{csch} x \, dx = 2 \int_{-\infty}^\infty x \, dx/(e^x - e^{-x}) = 4 \int_0^\infty x \, dx/(e^x - e^{-x})$$

$$= 4 \int_0^\infty xe^{-x} \, dx/(1 - e^{-2x}) = 4 \sum_{j=1}^\infty (2j - 1)^{-2}$$

$$\cdot \int_0^\infty (2j - 1)^2 \, xe^{-(2j-1)x} \, dx = 4\zeta_u(2)\Gamma(2)$$

$$= 4(\pi^2/8)(1) = \pi^2/2.$$ (See F9.C.)

$$\text{F24.}\quad C = \int_0^\infty e^{-az} \sinh(bz)^{1/2}\, dz = \frac{1}{b}\int_0^\infty \left(e^{-\frac{a}{b}\zeta^2+\zeta} - e^{-\frac{a}{b}\zeta^2-\zeta}\right) \zeta\, d\zeta \qquad \begin{matrix}(bz = \zeta^2)\\(\zeta = -\eta)\end{matrix}$$

$$= \frac{1}{b}\left\{-\int_{-\infty}^{0} e^{-\frac{a}{b}\eta^2-\eta}\,\eta\, d\eta - \int_0^\infty e^{-\frac{a}{b}\zeta^2-\zeta}\,\zeta\, d\,\zeta\right\}$$

$$= -\frac{1}{b}\int_{-\infty}^{\infty} e^{-\frac{a}{b}\left(\eta+\frac{b}{2a}\right)^2+\frac{b}{4a}}\,\eta\, d\eta \qquad \left(\eta + \frac{b}{2a}\right) = \xi$$

$$= (-e^{b/4a}/b)\int_{-\infty}^{\infty} e^{-\frac{a}{b}\xi^2}\left(\xi - \frac{b}{2a}\right) d\xi$$

$$= (e^{b/4a}/b)(b/2a)(b/a)^{1/2}\int_{-\infty}^{\infty} e^{-\frac{a}{b}\xi^2}\, d(\sqrt{a/b}\;\xi)$$

$$= (e^{b/4a}b^{1/2}/2a^{3/2}) \cdot 2 \cdot (1/2)\Gamma(1/2) = e^{b/4a}(b\pi)^{1/2}/2a^{3/2}.$$

A Note on Statistics

In judging the reliability of a sampling device, the following test may be useful. For a discrete density $p(\nu)$, precompute $p_i = p(i)$ for any desired set of argument values $\nu = i$. For a continuous density $p(\nu)$, compute

$$p_i = \int_{a_i}^{b_i} p(\nu)\, d\nu$$

for a suitable set of intervals (a_i,b_i). Let the density $p(\nu)$ be sampled for ν a large number N of times, according to the rule adopted, and tally the number f_i of times the sample results in $\nu = i$, or $\nu \in (a_i,b_i)$. Fixing attention on any <u>one</u> such index i, this may be regarded as a Bernoulli sequence of N trials, with p_i the probability of "success," $q_i = 1 - p_i$ the probability of "failure" in any one trial, and f_i the total number of successes.

In this situation, the law of large numbers states that

$$P_N = P\{|(f_i/N) - p_i| < \rho p_i\} \geq 1 - (q_i/\rho^2 p_i N) \to 1,$$

while the central limit theorem asserts that

$$P_N = \Phi((\rho^2 p_i N/q_i)^{1/2}) + Z_N$$

where

$$\Phi(z) = (1/2\pi)^{1/2} \int_{-z}^{z} e^{-\zeta^2/2}\, d\zeta, \text{ and } Z_N \to 0.$$

Note that the statistical reliability of a rule, insofar as it is correct, depends only on the density $p(\nu)$, and not on the rule. On the basis of the central limit theorem, the chance of a relative error $< \rho_1 \equiv (q_i/p_i N)^{1/2}$ is $\simeq \Phi(1) = .6826$, while the chance of a relative error $< 2\rho_1$, is $\simeq \Phi(2) = .9544$.

D-INDEX

Discrete Densities

D1. $p(\nu)$. General discrete.

D2. $e^{-\xi}\xi^{\nu}/\nu!$. Poisson.

D3. λ^j/j . Log series. [22, v.1; p. 166]

D4. $1/j^n$. Zeta, Zipf-Estoup, word distribution. [22, v.1; p. 240]

D5. $P\{f(\nu) = u\}$. Density for value of a function.

D6. $\prod_1^n p_i(\nu_i)$. Vector density.

D7. $P\{f(\nu_1, \ldots, \nu_n) = u\}$. Density for value of a function.

D8. $\binom{n}{s} q^{n-s} p^s;\ s = 0,1, \ldots, n$. Binomial, drawing with replacement.

D9. $\binom{n}{s} M(M + a) \ldots (M + (s - 1)a) \cdot N(N + a) \ldots (N + (n - s - 1)a)$. Polya's Urn.

D10. $\sum_a^b \binom{M}{\mu}\binom{N}{\mu - k} q^{M+N+k-2\mu} p^{2\mu-k}$. Binomial difference. [22, v.1; p. 55]

D11. $\binom{n-1}{s-1} q^{n-s} p^s;\ n = s, s + 1, \ldots$. Negative binomial.

D12. $q^{n-1}p$. Geometric.

D13. $a^{n-1}/(1 + a)^n$. Pascal.

D14. $(\alpha/(1 + \alpha\beta))^d \cdot (1 + \beta) \ldots (1 + (d - 1)\beta)/d!$. Polya.

D15. $\binom{d + s - 1}{s - 1} q^d p^s$. Failures before s-th success.

D16. $q^d p$. Failures before 1st success.

D17. $\binom{M}{s}\binom{N}{n - s}$. Hypergeometric, drawing without replacement.

D18.	$\left[\binom{M}{s-1}\binom{N}{n-s}\Big/\binom{M+N}{n-1}\right] \cdot \left[(M-(s-1))/((M+N)-(n-1))\right]$.	Negative hypergeometric. [22, v.1; p. 157]
D19.	$1/N(N-1)\ \ldots\ (N-n+1)$.	Random permutation.
D20.	$1\Big/\binom{N}{n}$.	Random combination.
D21.	$\{1 - 1/1! + 1/2! - \ldots + (-1)^{N-k}/(N-k)!\}/k!$.	k coincidences, matching. [22, v.1; p. 264]
D22.	$(n!/\mu_1!\ \ldots\ \mu_f!)\ p_1^{\mu_1} \ldots p_f^{\mu_f}$.	Multinomial, macrostate, particles in boxes.
D23.	$\binom{f}{k}\sum_{0}^{k}(-1)^i\binom{k}{i}(k-i)^n/f^n$.	Arfwedson, occupancy. [22, v. 1; p. 251]
D24.	$\sum_{j} f(j,k)$.	Discrete-discrete marginal.
D25.	$p^k\sum_{j\geq k/n}\binom{nj}{k}q^{nj-k}\lambda^j/j!$.	Poisson-compounded binomial. [22, v.1; p. 190]
D26.	$\sum_{j} p(j)f_j(k)$.	$p(j)$-compounded $f_j(k)$ density.
D27.	$(\phi^k/k)\sum_{j=1}^{\infty} j^{k-1}(\lambda e^{-\phi})^j$.	Log series-compounded Poisson. [22, v.1; p. 211]
D28.	$(e^{-\lambda}/k!)\sum_{0}^{\infty}(\lambda e^{-\phi})^j(\phi j)^k/j!$.	Neyman type A, contagious, Poisson-compounded Poisson. [22, v.1; p. 217]
D29.	$\begin{cases} e^{-\lambda(1-p^K)}; & k=0 \\ \sum_{1}^{\infty}\left[(e^{-\lambda}q^k/k!)\Gamma(Kj+k)(\lambda p^K)^j \cdot (\Gamma(Kj)j!)^{-1}\right]; & k=1,2,\ldots \end{cases}$	Poisson-compounded negative binomial. [22, v.1; p. 196]

D30. $\begin{cases} e^{-\lambda}; \; k = 0 \\ \displaystyle\sum_{1 \le j \le k/N} e^{-\lambda} \binom{k-1}{Nj-1} (\lambda p^N)^j q^{k-Nj}/j! \; ; \\ \qquad k = N, N+1, \ldots \; . \end{cases}$ Generalized Polya-Aeppli. [22, v.1; p. 197]

D31. $\begin{cases} e^{-\lambda}; \; k = 0 \\ \displaystyle\sum_{j=1}^{k} e^{-\lambda} \binom{k-1}{j-1} (\lambda p)^j q^{k-j}/j! \; ; \\ \qquad k = 1, 2, \ldots \; . \end{cases}$ Polya-Aeppli. [22, v.1; p. 197]

D32. $\displaystyle\sum_{0}^{N} \binom{N}{j} (\phi j)^k (pe^{-\phi})^j q^{N-j}/k! \; .$ Binomial-compounded Poisson. [22, v.1; p. 186]

D33. $\displaystyle\int_a^b dx \, f(x,k) \; .$ Continuous discrete marginal.

D34. $\displaystyle\int_0^{\xi} x^k e^{-x} \, dx/k!\xi$ Residual Poisson, Poisson's exponential, binomial limit. [22, v.1; p. 262]

$$= \sum_{\nu=k+1}^{\infty} e^{-\xi} \xi^{\nu-1}/\nu! \; .$$

D35. $\displaystyle\int_a^b dx \, p(x) f_x(k) \; .$ p(x)-compounded $f_x(k)$ density.

D36. $\displaystyle e^{-a} \sum_{0}^{k} a^{\nu}/\nu! - e^{-b} \sum_{0}^{k} b^{\nu}/\nu! \; .$ Uniform-compounded Poisson. [22, v.1; p. 184]

D37. $q^k p^s \Gamma(s+k)/\Gamma(s)k! \; .$ Negative binomial, $s > 0$ arbitrary, Γ-compounded Poisson. [22, v.1; p. 125]

D38. $\binom{N}{k} B(k+a, N-k+b) \; .$ B-compounded binomial. [22, v.1; p. 79]

D39. $\Gamma(s+k) B(a+s, b+k)/\Gamma(s)k! \; .$ B-compounded negative binomial.

D40. $B(a + 1, 1 + k)$. Simon, power-compounded geometric. [22, v.1; p. 245]

D41. $\int_0^\infty e^{-(A+B)u}(1 - e^{-Bu})^k \, du$. Yule, exponential-compounded geometric. [22, v.1; p. 245]

D42. $1/(e^{a+b}-1)$. A density for fractions a/b. [22, v.1; p. 31]

D43. $\prod_1^\infty p_i(\nu_i)$. Random sequences of integers.

Discrete Densities

D1. $p(\nu)$; $\nu = 0,1,2, \ldots$.

R_x. Set $\nu = \min\left\{k; \sum_0^k p(\nu) \geq r_0\right\}$.

J. $p(k) = \sum_0^k p(\nu) - \sum_0^{k-1} p(\nu)$ is the probability for $\nu = k$.

The continuous analogue in C1 is $\int_a^\nu p(\nu)\, d\nu = r_o$.

Note. In this and other densities, the obvious changes required for other domains of the variable are left to the reader.

D2. $p(\nu) = e^{-\xi}\xi^\nu/\nu!$; $\nu = 0,1,2, \ldots, \xi > 0$.

R_x1. Set $\nu = \min\left\{k; \sum_0^k \xi^\nu/\nu! \geq r_0 e^\xi\right\}$.

J1. Special case of D1. (see F3.)

R_x2. Set $\nu = -1 + \min\{n; \hat{r}_1\hat{r}_2 \ldots \hat{r}_n < e^{-\xi}\}$.

J2. Note that $P_n \equiv P\{r_1 \ldots r_n < e^{-\xi}\} = 1 - P\{r_1 \ldots r_n \geq e^{-\xi}\}$

$= 1 - \int_{\left\{r_1 \ldots r_n \geq e^{-\xi}\right\}} dr_1 \ldots dr_n$. Making the transformation $r_i = e^{-\nu_i}$,

$0 < \nu_i < \infty$, this becomes $P_n = 1 - \int\limits_{\left\{\begin{array}{c}\Sigma \nu_i \le \xi \\ \nu_i > 0\end{array}\right\}} \prod_1^n e^{-\nu_i}\, d\nu_i$.

It is shown in C45 that the latter integral, namely the distribution function for the sum $u = \sum_1^n \nu_i$ under the density $e^{-\nu_1} \ldots e^{-\nu_n}$, has the value $\int_0^\xi u^{n-1} e^{-u}\, du/(n-1)!$. Hence $P_n = \int_\xi^\infty u^{n-1} e^{-u}\, du/(n-1)!$ $= \sum_0^{n-1} e^{-\xi} \xi^\nu/\nu!$, the Poisson distribution. (F3A.)

If p_n is the probability that, in a sequence $\hat{r}_1, \hat{r}_2, \ldots$ of random numbers, n is the first index for which $\hat{r}_1 \ldots \hat{r}_n < e^{-\xi}$, then $P\left\{r_1 \ldots r_{n+1} < e^{-\xi}\right\} = P\left\{r_1 \ldots r_n < e^{-\xi}\right\} + p_{n+1}$. So by the preceding result, we have $\sum_0^n p(\nu) = \sum_0^{n-1} p(\nu) + p_{n+1}$. Hence $p(n) = p_{n+1}$, and the rule follows. Note that $p_1 = e^{-\xi} = p(0)$.

R_x3. (For large ξ.) Sample the density $e^{-y^2/2}/(2\pi)^{1/2}$ for y on $(-\infty,\infty)$ by C60 or R9. If $\xi - 1/2 + y\xi^{1/2} < 0$, set $\nu = 0$. Otherwise let ν be the nearest integer to $\xi - 1/2 + y\xi^{1/2}$.

J3. For large ξ, the Poisson distribution is approximately normal, namely

$$P(\nu) = \sum_0^\nu e^{-\xi} \xi^\nu/\nu! \approx (1/2\pi)^{1/2} \int_{-\infty}^{(\nu + 1/2 - \xi)/\xi^{1/2}} e^{-\zeta^2/2}\, d\zeta \ .$$

[21; p. 717.] The rule results from setting $(\nu + 1/2 - \xi)/\xi^{1/2} = y$.

D3. $p(j) = \lambda^j/jL(\lambda)$; $j = 1,2, \ldots, 0 < \lambda < 1$, $L(\lambda) = -\ell n(1 - \lambda)$.

R_x. Set $j = \min\left\{k;\ \sum_1^k \lambda^j/j \geq r_o L(\lambda)\right\}$.

J. Noting that $L(\lambda) = \sum_1^\infty \lambda^j/j$, the rule follows from D1.

D4. $q(j) = 1/j^n\zeta(n)$; $j = 1, 2, \ldots, n > 1$, $\zeta(n)$ the ζ-function of F9.

R_x. Set $j = \min\left\{k;\ \sum_1^k 1/j^n \geq r_o\ \zeta(n)\right\}$.

J. The rule follows from D1.

D5. $q(u) = P\{f(\nu) = u\} = \sum_{\{\nu;f(\nu)=u\}} p(\nu)$; $p(\nu)$ discrete density for $\nu = 0,1,2, \ldots$, $f(\nu)$ defined for $\nu = 0,1,2, \ldots$.

R_x. Sample $p(\nu)$ for ν. Set $u = f(\nu)$.

J. $q(u)$ is the density for the value u of the function $f(\nu)$.

D6. $p(\nu) = p(\nu_1, \ldots,\nu_n) = \prod_1^n p_i(\nu_i)$; $p_i(\nu_i)$ density for ν_i.

R_x. Sample each $p_i(\nu_i)$ for ν_i. Set vector $\nu = (\nu_1, \ldots, \nu_n)$.

J. $p(\nu)$ is the density for a vector whose components are independent.

D7. $q(u) = P\{f(\nu_1, \ldots,\nu_n) = u\} = \sum_{\{\nu;f(\nu)=u\}} p_1(\nu_1) \ldots p_n(\nu_n)$; $p_i(\nu_i)$ discrete density for ν_i, $f(\nu)$ defined for $\nu = (\nu_1, \ldots,\nu_n)$.

R_x. Sample each $p_i(\nu_i)$ for ν_i. Set $u = f(\nu_1, \ldots, \nu_n)$.

J. $q(u)$ is the density for the value u of the function $f(\nu_1, \ldots, \nu_n)$ under the density $p_1(\nu_1) \ldots p_n(\nu_n)$.

D8. $q(s) = \binom{n}{s} q^{n-s} p^s$; $s = 0,1, \ldots, n$, $0 < p < 1$, $q = 1 - p$.

R_x. Set s = number of random numbers $r_1, \ldots, r_n$ for which $r_i \leq p$.

J. For $\nu_i \varepsilon \{0,1\}$, $p_i(\nu_i) = \begin{cases} q \text{ for } \nu_i = 0 \\ p \text{ for } \nu_i = 1 \end{cases}$, $i = 1, \ldots, n$, the function $f(\nu) = \nu_1 + \ldots + \nu_n$, under the density $p_1(\nu_1) \ldots p_n(\nu_n)$, has the probability $P\{f(\nu) = s\} = \sum_{\{\nu_1+\ldots+\nu_n=s\}} p_1(\nu_1) \ldots p_n(\nu_n)$

$= q^{n-s}p^s \sum_{\{\nu_1+\ldots+\nu_n=s\}} 1 = \binom{n}{s} q^{n-s}p^s$. The rule follows from D7.

Note. The binomial density q(s) is the probability of s "successes" in n trials of an elementary event for which p is the chance of success ($\nu_i = 1$). In the urn model, this means drawing "with replacement." Thus, if p and q are respectively the chances of drawing a white or or black ball, then $p^s q^{n-s}$ is the chance that, in n successive drawings with replacement, exactly s balls should be white, while $\binom{n}{s}$ is the number of ways in which the s white balls could appear.

D9. $p(s) = \binom{n}{s}\dfrac{M(M+a)\ldots(M+(s-1)a)\cdot N(N+a)\ldots(N+(n-s-1)a)}{S(S+a)\qquad\ldots\qquad(S+(n-1)a)};$

$s = 0,1,\ldots,n$, $S = M + N$, an integer ≥ 0.

R_X. One follows the steps:

1. Put $M \to \hat{M}$, $S \to \hat{S}$, $0 \to s$, $1 \to t$.
2. If $r_t \leq \hat{M}/\hat{S}$, put $\hat{M} + a \to \hat{M}$, $s + 1 \to s$. Go to (3).

   If $r_t > \hat{M}/\hat{S}$, go to (3).
3. If $t = n$, exit with s. If $t < n$, put $t + 1 \to t$, $\hat{S} + a \to \hat{S}$, and return to (2).

J. p(s) is the probability of drawing s white balls in n successive drawings from an urn initially containing M white and N black balls, subject to the condition:

(C) On the t-th drawing, the ball drawn is replaced, and a more balls, of the same color as that drawn, are added to the urn.

Note 1. The second factor of p(s) is the chance of a drawing in which any designated set of s positions are white, and $\binom{n}{s}$ is the number of ways in which the white balls might appear.

Note 2. For a = 0, $p(s) = \binom{n}{s}M^sN^{n-s}/S^n = \binom{n}{s}(N/S)^{n-s}(M/S)^s$ is the binomial density of D8.

D10. $p(k) = \sum_{\mu=a}^{b} \binom{M}{\mu}\binom{N}{\mu-k} q^{M+N+k-2\mu} p^{2\mu-k}; \; -N \leq k \leq M,$

$a = \max\{0,k\}$, $b = \min\{M, N+k\}$, $0 < p < 1$, $q = 1 - p$.

R_x. Set μ = number of $r_1, \ldots, r_M$ such that $r_i \leq p$.

Set ν = number of $r'_1, \ldots, r'_N$ such that $r'_j \leq p$. Set $k = \mu - \nu$.

J. $p(k)$ is the probability that the function $\mu - \nu$ have value k, $-N \leq k \leq M$, where μ and ν have respectively the binomial densities

$$p_1(\mu) = \binom{M}{\mu} q^{M-\mu} p^{\mu};\ \mu = 0, 1, \ldots, M,$$

$$p_2(\nu) = \binom{N}{\nu} q^{N-\nu} p^{\nu};\ \nu = 0, 1, \ldots, N.$$

$$\text{For,}\ \sum_{\mu-\nu=k} p_1(\mu)p_2(\nu) = \sum_{\mu=0}^{M} \binom{M}{\mu} q^{M-\mu} p^{\mu} \binom{N}{\mu-k} q^{N-(\mu-k)} p^{\mu-k}$$

$$= \sum_{\mu=0}^{M} \binom{M}{\mu}\binom{N}{\mu-k} q^{M+N+k-2\mu} p^{2\mu-k},$$

where necessarily $0 \leq \mu \leq M$, and $0 \leq \mu - k \leq N$, i.e.,

$$0, k \leq \mu \leq M, N + k.$$

This accounts for the limits a,b in the above sum on μ. The rule then follows from D7 and D8.

<u>D11. $p(n) = \binom{n-1}{s-1} q^{n-s} p^s$</u>; $n = s, s+1, \ldots$, $s \geq 1$ fixed, $0 < p < 1$, $q = 1 - p$.

R_x. Set n = first m for which s of the random numbers $r_1, \ldots, r_m$ are $\leq p$.

J. $p(n) = \left\{\binom{n-1}{s-1} q^{(n-1)-(s-1)} p^{s-1}\right\} p$ is the probability of exactly s "successes" occurring for the first time on the n-th trial.

<u>Note 1.</u> $1 = (1-q)^{-s} p^s = \sum_{d=0}^{\infty} \frac{(-s)(-s-1)\ldots(-s-d+1)}{d!} (-q)^d p^s$

$$= \sum_{d=0}^{\infty} \frac{(s+d-1)\ldots(s+1)(s)}{d!} q^d p^s = \sum_{d=0}^{\infty} \binom{s+d-1}{d} q^d p^s$$

$$= \sum_{n=s}^{\infty} \binom{n-1}{n-s} q^{n-s} p^s = \sum_{n=s}^{\infty} \binom{n-1}{s-1} q^{n-s} p^s = \sum_{n=s}^{\infty} p(n).$$

This verifies that p(n) is indeed a density and accounts for the term "negative binomial."

Note 2. For an alternative rule, see D12, Note.

Note 3. $p(n) < p\,(n+1)$ iff $n < (s-1)/p$.

D12. $g(n) = q^{n-1}p$; $n = 1,2,3, \ldots, 0 < p < 1, q = 1 - p$.

R_x1. Set $n = \min\{m;\ r_m \leq p\}$.

J1. Case s = 1 of D11. g(n) is the probability of the first success occurring on the n-th trial.

R_x2. Set $n = k$ where k is defined by $k - 1 < \ell n\, r_1/\ell n\, q \leq k$, $(k \geq 1)$.

J2. The rule follows from D1, which would set n = k, where

$$\sum_{1}^{k-1} q^{\nu-1}p < r_o \leq \sum_{1}^{k} q^{\nu-1}p \quad , \text{ i.e., } 1 - q^{k-1} < r_o \leq 1 - q^k \quad ,$$

or, with $r_1 = 1 - r_o$, $q^k \leq r_1 < q^{k-1}$.

Equivalently, $k\,\ell n\, q \leq \ell n\, r_1 < (k-1)\ell n\, q$.

Since both logs are negative, the rule follows.

Note. If each of s independent variables n_i has the density $g(n) = q^{n-1}p$, $n = 1,2,3, \ldots$ of D12, then the probability that their sum $n_1 + \ldots + n_s$ have the value n under the product of these s densities is

$$P\{n_1 + \ldots + n_s = n\} = \sum_{\{n_1+\ldots+n_s=n\}} \left(q^{n_1-1}p\right) \ldots \left(q^{n_s-1}p\right)$$

$$= \sum_{\{n_1+\ldots+n_s=n\}} q^{n-s}p^s = \binom{n-1}{s-1} q^{n-s}p^s.$$

For, the number of terms in the final sum is the number of vectors $(n_1,\ldots,n_s)$, for which $n_1 + \ldots + n_s = n$, $n_i \geq 1$ (compositions), and this is well known to be $\binom{n-1}{s-1}$, namely, the number of ways of choosing s − 1 partition places out of n − 1 possible places 1, 2, ..., n − 1.

Hence the density of D11 may be regarded as the density for the sum of s variables ν_i, each with the density of D12. Thus, by D7 and D12, R_x2., we have for D11 the alternative rule:

$$\text{Set } n = \sum_{1}^{s} k_j, \text{ where } k_j - 1 < \ln r_j/\ln q \leq k_j \quad .$$

D13. $q(n) = a^{n-1}/(1 + a)^n$; $n = 1, 2, 3, \ldots, a > 0$.

R_x1. Set $n = \min \{m; r_m \leq 1/(1 + a)\}$.

R_x2. Set $n = k$, where $k - 1 < \ln r_1/(\ln a - \ln(1 + a)) \leq k$.

J. Special case of D12, with $p = 1/(1 + a)$.

D14. $h(d) = (1 + \alpha\beta)^{-1/\beta}(\alpha/(1 + \alpha\beta))^d \, 1 \cdot (1 + \beta) \ldots (1 + (d - 1)\beta)/d!$;

$d = 0,1,2, \ldots, h(0) \equiv (1 + \alpha\beta)^{-1/\beta}, \alpha > 0, \beta \in \{1,1/2,1/3, \ldots\}$.

R_x. Define $s = 1/\beta \in \{1,2,3, \ldots\}$, $p = 1/(1 + \alpha\beta)$. Set $d = -s +$ (first n for which s of the random numbers $r_1, \ldots, r_n$ are $\leq p$).

J. Special case of D11, with $p = 1/(1 + \alpha\beta)$, $q = \alpha\beta/(1 + \alpha\beta)$, $s = 1/\beta$. In fact, $h(d) = (1 + \alpha\beta)^{-1/\beta}(\alpha\beta/(1 + \alpha\beta))^d(1/\beta)(1/\beta + 1) \ldots$

$$\cdot (1/\beta + (d - 1))/d!$$

$$= p^s q^d (d - 1 + s) \ldots (s + 1)(s)/d! = \binom{d + s - 1}{d} q^d p^s$$

$$= \binom{d + s - 1}{s - 1} q^d p^s = \binom{n - 1}{s - 1} q^{n-s} p^s, \text{ where } n = d + s.$$

The rule follows from D11. Note that $h(0) = p^s$, as required.

Note. See D12, Note for another rule for D11.

D15. $q(d) = \binom{d + s - 1}{s - 1} q^d p^s$; $d = 0,1,2, \ldots$, s integer ≥ 1, $0 < p < 1$,

$q = 1 - p$.

R_x. Set n = first m for which s of the random numbers $r_1, \ldots, r_m$ are $\leq p$, and $d = n - s$.

J. For $d = n - s$, one has $q(d) = \binom{n - 1}{s - 1} q^{n-s} p^s$; $n = s, s + 1, \ldots$, as in D11.

<u>Note.</u> $q(d)$ is the probability of d failures before the s-th success.

D16. $q(d) = q^d p$; $d = 0,1,2, \ldots, 0 < p < 1, q = 1 - p$.

R_x1. Set $d = -1 +$ (first m for which $r_m \leq p$).

R_x2. Set $d = -1 + k$, where $k - 1 < \ln r_1/\ln q \leq k$.

J. For $d = n - 1$, one has $q(d) = q^{n-1} p$, $n = 1,2, \ldots$, as in D12.

<u>Note.</u> $q(d)$ is the probability of d failures before the first success.

D17. $p(s) = \binom{M}{s}\binom{N}{n-s}\Big/\binom{M+N}{n}$; $\max\{0, n-N\} \le s \le \min\{n, M\}$, $1 \le n \le M+N$.

R_x. One follows the steps:

1. Put $M \to \hat{M}$, $N \to \hat{N}$, $M + N \to \hat{S}$, $0 \to s$, $1 \to t$.
2. If $r_t \le \hat{M}/\hat{S}$, put $\hat{M} - 1 \to \hat{M}$, $s + 1 \to s$. Go to (3).

   If $r_t > \hat{M}/\hat{S}$, put $\hat{N} - 1 \to \hat{N}$. Go to (3).
3. If $t = n$, exit with s. If $t < n$, go to (4).
4. If $\hat{M} = 0$, exit with $s(=M)$. If $\hat{M} > 0$, go to (5).
5. If $\hat{N} = 0$, put $(n - t) + s \to s$, exit with s. If $\hat{N} > 0$, go to (6).
6. Put $t + 1 \to t$, $\hat{S} - 1 \to \hat{S}$, and return to (2).

J. p(s), as written above, is obviously the probability of obtaining s white balls in a choice of n balls from an urn containing M white and N black balls. If we write

$$P_s^M = M(M-1) \ldots (M-s+1) = M!/(M-s)!$$

for the number of permutations of M things taken s at a time, it is easy to show that

$$p(s) = \left(P_s^M P_{n-s}^N \Big/ P_n^{M+N}\right)\binom{n}{s}$$

, which is the chance of drawing s white balls in n consecutive drawings from the urn, without replacement. The rule is based on the latter interpretation.

Note that $P_s^M P_{n-s}^N \Big/ P_n^{M+N}$ is the chance of such a drawing in which the first s balls are white, while $\binom{n}{s}$ is the number of possible orders in which the s white balls might appear.

Note. In step (5) of the rule, if $\hat{N} = 0$, then $t = s + N$, and hence for $s' \equiv n - t + s = n - N \le M$, one has $s' \le M$, as it must be.

D18. $q(n) = \dfrac{\binom{M}{s-1}\binom{N}{n-s}}{\binom{M+N}{n-1}} \cdot \dfrac{M-(s-1)}{(M+N)-(n-1)}$; $n = s, s+1, \ldots, s+N$,

M,N,s integers > 0, s fixed, $1 \le s \le M$.

R_x. One follows the steps:

1. Put $M \to \hat{M}$, $M + N \to \hat{S}$, $0 \to \sigma$, $1 \to t$.
2. If $r_t \le \hat{M}/\hat{S}$, put $\hat{M} - 1 \to \hat{M}$, $\sigma + 1 \to \sigma$. Go to (3).

   If $r_t > \hat{M}/\hat{S}$, go to (3).
3. If $\sigma < s$, put $\hat{S} - 1 \to \hat{S}$, $t + 1 \to t$. Return to (2).

   If $\sigma = s$, exit with $n = t$.

J. q(n) is the probability that the n-th drawing without replacement from an urn containing M white and N black balls should produce a total of exactly s white balls for the first time. For, by D17, the first fraction above is the chance that the first n − 1 draws should produce exactly s − 1 white balls, while the second fraction is the chance that the next (n-th) draw should then be white.

Note. This is the "without replacement" analogue of the negative binomial density D11.

D19. $p(\Pi) = 1/N(N - 1) \ldots (N - n + 1)$; Π ranging over the P_n^N equally likely permutations $\Pi = (C_1, \ldots, C_n)$ of the integers 1, 2, ..., N, taken n at a time.

R_x. One follows the steps.

1. List the integers 1, ..., N.
2. Put $N \to \hat{N}$, $1 \to t$.
3. Set $K = \min \{k; k \ge \hat{N} r_t\}$, $K \in \{1, \ldots, \hat{N}\}$.
4. Set C_t = K-th integer of the remaining list, and delete this integer from the remaining list.
5. If $t < n$, put $1 + t \to t$, $\hat{N} - 1 \to \hat{N}$, and return to (3).

   If $t = n$, exit with permutation $\Pi = (C_1, \ldots, C_n)$.

J. Obvious.

D20. $p(C) = 1\Big/\binom{N}{n}$; C ranging over the $\binom{N}{n}$ equally likely combinations $C = \{C_1, \ldots, C_n\}$ of the integers 1, ..., N, taken n at a time.

R_x. Obtain the random permutation $\Pi = (C_1, \ldots, C_n)$ from D19.
Let C be the unordered set $\{C_1, \ldots, C_n\}$.

J. The $P_n^N = N!/(N - n)!$ equally likely permutations of D19 may be partitioned into $\binom{N}{n} = N!/n!(N - n)!$ classes, each containing the same number n! of permutations. The classes are therefore also equally likely.

D21. $p(k) = (1/k!)\{1 - 1/1! + 1/2! - \dots + (-1)^{N-k}/(N-k)!\}$;

$k = 0,1, \dots, N$.

R_x. One follows the steps:

1. List the integers $1,2, \dots, N$.
2. Put $N \to \hat{N}$, $1 \to t$.
3. Set $K = \min\{k, k \geq \hat{N}r_t\}$, $K \in \{1,2, \dots, \hat{N}\}$.
4. Set C_t = K-th integer of the remaining list, and delete this integer from the remaining list.
5. If $t < N$, put $t + 1 \to t$, $\hat{N} - 1 \to \hat{N}$, and return to (3). If $t = N$, go to (6) with random permutation $\Pi = (C_1, \dots, C_N)$.
6. Set k = number of integers i for which $C_i = i$, $k = 0,1, \dots, N$.

J. The rule is the obvious adaptation of that in D19, since $p(k)$ is the probability of exactly k coincidences (fixed points) in a random permutation of the integers $1, \dots, N$. This may be seen from the inclusion-exclusion principle of F20 as follows. Consider first a set of integers $1, 2, \dots, n$, and let S_i be the set of all their permutations which leave the integer i fixed. By F20, the number of permutations which leave at least one integer fixed is

$$\#(S_1 \cup \dots \cup S_n) = \sum_{\binom{n}{1}} \#S_{i_1} - \sum_{\binom{n}{2}} \#S_{i_1}S_{i_2} + \dots + (-1)^{n-1} \#S_1 \dots S_n$$

$$= (n!/1!(n-1)!)(n-1)! - (n!/2!(n-2)!)(n-2)! + \dots$$

$$+ (-1)^{n-1} (n!/n!0!)(0)! = n!(1/1! - 1/2! + \dots$$

$$+ (-1)^{n-1}/n!).$$

Hence the number of permutations which leave no integer fixed is $n!(1 - 1/1! + 1/2! - \dots + (-1)^n/n!)$. Now, the number of permutations of of the original set $1, \dots, N$ which leave exactly k integers fixed is $\binom{N}{k}$ times the number which leave any particular choice of k integers fixed, with the remaining $N - k$ all unfixed. It follows from above that

$$p(k) = (1/N!)(N!/k!(N-k)!)(N-k)!(1 - 1/1! + 1/2! - \dots$$
$$+ (-1)^{N-k}/(N-k)!) = 1/k!\,\{1 - 1/1! + 1/2! - \dots$$
$$+ (-1)^{N-k}/(N-k)!\}.$$

D22. $p[\mu_1, \dots, \mu_f] = (n!/\mu_1! \dots \mu_f!)p_1^{\mu_1} \dots p_f^{\mu_f}$; domain: all "multiplicity" vectors $[\mu_1, \dots, \mu_f]$ with $\mu_j \geq 0$ and sum

$$\sum_1^f \mu_j = n;\ p_j > 0,\ \sum_1^f p_j = 1.$$

R_x. One follows the steps:

1. Put $0 \to \mu_1, \dots, 0 \to \mu_f$; $1 \to t$.
2. Set $K = \min\,\{k; \sum_1^k p_j \geq r_t\}$. Put $\mu_K + 1 \to \mu_K$.
3. If $t < n$, put $t + 1 \to t$ and return to (2). Otherwise exit with $[\mu_1, \dots, \mu_f]$.

J. $p[\mu_1, \dots, \mu_f]$ is the probability that a vector $(\nu_1, \dots, \nu_n)$ should have μ_1 components 1, ..., μ_f components f (multiplicities), where each component ν_i has probability p_j of value j, $j = 1, \dots, f$.

Note 1. We may partition the f^n vectors $(\nu_1, \dots, \nu_n)$ into "multiplicity classes" $C[\mu_1, \dots, \mu_f]$, μ_j of the components ν_i having value j. Such a class necessarily has $\mu_1 + \dots + \mu_f = n$, $\mu_j \geq 0$. The number of such classes is easily shown to be $\binom{n+f-1}{f-1}$; this is the number of vectors $[\mu_1, \dots, \mu_f]$ in the domain of $p[\mu_1, \dots, \mu_f]$. Moreover, the number of vectors $(\nu_1, \dots, \nu_n)$ belonging to a particular class

$$C[\mu_1, \dots, \mu_f] \text{ is } \binom{n}{\mu_1}\binom{n-\mu_1}{\mu_2}\dots\binom{n-\mu_1-\dots-\mu_{f-1}}{\mu_f} = n!/\mu_1! \dots \mu_f!,$$

each such vector $(\nu_1, \dots, \nu_n)$ having probability $p_1^{\mu_1} \dots p_f^{\mu_f}$. The probability of the class $C[\mu_1, \dots, \mu_n]$ is therefore

$(n!/\mu_1! \dots \mu_f!)\ p_1^{\mu_1} \dots p_f^{\mu_f}.$

Note that algebraically

$$1 = (p_1 + \dots + p_f)^n = \sum_{(\nu_1,\dots,\nu_n)} p_{\nu_1} \dots p_{\nu_n}$$

$$= \sum_{C[\mu_1,\dots,\mu_f]} \ \sum_{(\nu_1,\dots,\nu_n)\varepsilon C[\mu_1,\dots,\mu_f]} p_{\nu_1} \dots p_{\nu_n}$$

$$= \sum_{C[\mu]} \sum_{\nu\varepsilon C[\mu]} p_1^{\mu_1} \dots p_f^{\mu_f} = \sum_{C[\mu]} (n!/\mu_1! \dots \mu_f!) p_1^{\mu_1} \dots p_f^{\mu_f}$$

$$= \sum_{C[\mu]} p[\mu].$$

Example. $p[\mu_1, \dots, \mu_f]$ is the probability of assigning n balls to f boxes, p_j being the probability of box j, in such a way that box 1 contains μ_1 balls, ..., box f contains μ_f balls. For equally likely boxes, $p[\mu_1, \dots, \mu_f] = (n!/\mu_1! \dots \mu_f!)(1/f)^n$.

Note 2. The density for the sum $u = \nu_1 + \dots + \nu_n$ is not easily expressed (although easily sampled). Only for the case $f = 2$ do we have a connection with the binomial density D8.

D23. $p(k) = \binom{f}{k} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\frac{k-i}{f}\right)^n$; $k = 1,2, \dots, \min\{f,n\}$.

R_x. One follows the steps:

1. Put $0 \to \mu_1, \dots, 0 \to \mu_f$; $1 \to t$.
2. Set $K = \min\{k;\ k \geq fr_t\}$. Put $1 + \mu_K \to \mu_K$.
3. If $t < n$, put $t + 1 \to t$, and return to (2). If $t = n$, go to (4).
4. Set k = number of positive components of vector $[\mu_1, \dots, \mu_f]$ and exit with k.

J. The rule is the obvious adaptation of that in D22, since p(k) is the probability of exactly k of f boxes being occupied, if n particles are assigned to f eqûally likely boxes ($p_j = 1/f$, $j = 1, \ldots, f$). This may be seen from the inclusion-exclusion principle of F20. Fix on any <u>one</u> of the $\binom{f}{k}$ possible choices of k boxes to be occupied, the rest vacant. Let S_i, $i = 1, \ldots, k$, be the set of all assignments of the n particles to these k boxes which leave the i-th of these boxes empty. Then $S_1 \cup \ldots \cup S_k$ is the set of all such assignments which leave <u>at least one</u> vacant. By F20, the number of these assignments is

$$\#(S_1 \cup \ldots \cup S_k) = \sum_{\binom{k}{1}} \#S_{i_1} - \sum_{\binom{k}{2}} \#(S_{i_1} S_{i_2}) + \ldots + (-1)^{k-1} \#S_1 \ldots S_k$$

$$= \binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \ldots + (-1)^{k-1}\binom{k}{k}(0)^n.$$

The set of <u>all</u> assignments of the n particles to these k boxes has cardinal k^n, so the number of assignments leaving <u>none</u> of these k boxes vacant is the difference

$$D \equiv k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n - \ldots + (-1)^k\binom{k}{k}(0)^n.$$

Since there are $\binom{f}{k}$ choices of the k boxes to be occupied, the totality of assignments of the n particles to the f boxes leaving exactly k of of boxes occupied is $\binom{f}{k}$ D. Finally, the total number of assignments of n particles to f boxes is f^n. Hence $p(k) = \binom{f}{k} D/f^n$, which is the formula given in D23.

<u>Question 1.</u> Is it true that

$$p(k) = \sum (n!/\mu_1! \ldots \mu_f!)(1/f)^n,$$

where the sum ranges over all vectors $[\mu_1, \ldots, \mu_f]$, such that $\mu_1 + \ldots + \mu_f = n$, $\mu_j \geq 0$, with exactly k positive components?

<u>Question 2.</u> If $f > n$, at most n boxes can be occupied. Is it true that the formula for p(k) in D23 is automatically zero if $k > n$?

D24. $q(k) = \sum_{j=J}^{\infty} f(j,k)$; $k = K, K+1, \ldots$, $f(j,k)$ density for $j \geq J$, $k \geq K$.

R_x. Sample the marginal density $p(j) = \sum_{k=K}^{\infty} f(j,k)$ for $j \geq J$. For this j, sample the j-dependent k-density $p(k|j) = f(j,k)/p(j)$ for $k \geq K$.

J. Consider the obvious relations:

1. $p(j) = \sum_k f(j,k)$
2. $p(k|j) = f(j,k)/p(j)$
3. $q(k) = \sum_j f(j,k)$
4. $p(j|k) = f(j,k)/q(k)$

From (3) and (2), we have $q(k) = \sum_j p(j)p(k|j)$, which gives the rule.

Moreover, (4), (3), (2) imply

$$p(j|k) = f(j,k) \Big/ \sum_j f(j,k) = p(k|j)p(j) \Big/ \sum_j p(k|j)p(j),$$

which is Bayes' theorem.

Note 1. The analogues for other domains of j, k, discrete or continuous are obvious.

Note 2. The idea in this and all related rules is that the given density $q(k)$ is difficult to sample, but can be recognized as the marginal k-density of a two variable density $f(j,k)$, and that the marginal j density $p(j) = \sum_k f(j,k)$, and the j-dependent k-density $p(k|j) = f(j,k)/p(j)$ are relatively easy to sample. Moreover, the necessary fact that $\sum_k q(k) = 1$ may then be verified from the relation

$$\sum_k q(k) = \sum_k \left(\sum_j p(j)p(k|j) \right) = \sum_j p(j) \sum_k p(k|j) = \sum_j p(j) = 1.$$

D25. $q(k) = e^{-\lambda} p^k \sum_{j \geq k/n} \binom{nj}{k} q^{nj-k} \lambda^j / j!$; $k = 0,1,2, \ldots, \lambda > 0, 0 < p < 1,$

$q = 1 - p, n \in \{1,2,3,\ldots\}$.

R_x. Sample $p(j) = e^{-\lambda}\lambda^j/j!$ for $j \in \{0,1,2, \ldots\}$ by D2. If $j = 0$, set $k = 0$. If $j \geq 1$, set k = number of $r_1, \ldots, r_{nj}$ such that $r_i \leq p$.

J. The function $f(j,k) = (e^{-\lambda}\lambda^j/j!)\binom{nj}{k} q^{nj-k} p^k$ is a doubly discrete density on the set of all lattice points (j,k) for which $k \geq 0$, $j \geq k/n$, equivalently, $j \geq 0$, $0 \leq k \leq nj$. Its marginal k-density is the $q(k)$ given above, whereas its marginal j-density is $p(j) = \sum_{k=0}^{nj} f(j,k)$ $= e^{-\lambda}\lambda^j/j!$ Moreover, for each $j \geq 0$, the j-dependent k-density is $p(k|j) = f(j,k)/p(j) = \binom{nj}{k} q^{nj-k} p^k$; $k = 0,1, \ldots, nj$. Since $p(k|0) = 1$, and for $j \geq 1$, $p(k|j)$ is the binomial density of D8, the rule (an obvious extension of D24) follows. A continuous analogue is C135.

D26. $q(k) = \sum_{j=J}^{\infty} p(j) f_j(k)$; $k = K, K+1, \ldots$, $p(j)$ density for $j = J,$

$J + 1, \ldots$, $f_j(k)$ density for $k = K, K + 1, \ldots$, for each $j \geq J$.

R_x. Sample $p(j)$ for $j \geq J$. For this j, sample $f_j(k)$ for $k \geq K$.

J. Corollary of D24, with $f(j,k) \equiv p(j)f_j(k)$. Note that $p(j)$ is then the marginal j-density, and $p(k|j) = f_j(k)$.

D27. $q(k) = L^{-1}(\lambda)(\phi^k/k!) \sum_{j=1}^{\infty} j^{k-1}(\lambda e^{-\phi})^j$; $k = 0,1,2,\ldots, 0 < \lambda < 1, \phi > 0,$

$L(\lambda) = -\ln(1 - \lambda)$.

R_x. Sample $p(j) = \lambda^j/jL(\lambda)$ for $j \in \{1,2,3,\ldots\}$ by D3. For this j, sample $f_j(k) = e^{-j\phi}(j\phi)^k/k!$ for $k \in \{0,1,2,\ldots\}$ by D2.

J. One has $\sum_{j=1}^{\infty} p(j) f_j(k) = q(k)$ as given, and the rule follows from D26.

D28. $q(k) = (e^{-\lambda}/k!) \sum_{j=0}^{\infty} (\lambda e^{-\phi})^j (\phi j)^k / j!$; $k = 0,1,2, \ldots, \lambda, \phi > 0,$ and

(N.B.!) $(\phi j)^k \equiv 1$ for $j = 0, k = 0$ by definition.

R_x. Sample $p(j) = e^{-\lambda}\lambda^j/j!$ for $j \in \{0,1,2,\ldots\}$ by D2. If $j = 0$, set $k = 0$. For $j \geq 1$, sample $f_j(k) = e^{-\phi j}(\phi j)^k/k!$ for $k \in \{0,1,2, \ldots\}$ by D2.

J. We write $q(k) = \sum_{j=0}^{\infty} (e^{-\lambda}\lambda^j/j!)(e^{-\phi j}(\phi j)^k/k!)$ in the form of D26, where $p(j) = e^{-\lambda}\lambda^j/j!$ is a density for $j \in \{0,1,2,\ldots\}$, and $f_j(k) = e^{-\phi j}(\phi j)^k/k!$ is a density for $k \in \{0,1,2,\ldots\}$, for each such j. For $j \geq 1$, $f_j(k)$ is a Poisson density with parameter $\xi = \phi j > 0$, as in D2, whereas for $j = 0$, $f_j(k) = 1$ for $k = 0$, and 0 for all $k > 0$ by definition. The rule then follows from D26.

D29. $$q(k) = \begin{cases} e^{-\lambda(1 - p^K)}; & k = 0 \\ \sum_{j=1}^{\infty} (e^{-\lambda}q^k/k!)\Gamma(Kj + k)(\lambda p^K)^j/\Gamma(Kj)j!; & k = 1,2, \ldots, \end{cases} \quad \lambda, K > 0,$$

$0 < p < 1$, $q = 1 - p$.

R_x. Sample the Poisson density $e^{-\lambda}\lambda^j/j!$ for $j \in \{0,1,2, \ldots\}$ by D2. If $j = 0$, set $k = 0$. If $j \geq 1$, sample the negative binomial density $q^k p^{Kj}\Gamma(Kj + k)/\Gamma(Kj)k!$ for $k \in \{0,1,2, \ldots\}$ by D11 or D37, with $s = Kj$. (For K integral, obtain n from D11 and set $k = n - s$.)

J. We define $p(j) = e^{-\lambda}\lambda^j/j!$; $j = 0,1,2, \ldots$, and for every such j, we define the function $f_j(k)$ for $k \in \{0,1,2, \ldots\}$ by

$$f_j(k) = \begin{cases} 1 \text{ for } j = 0,\ k = 0;\ 0 \text{ for } j = 0,\ k = 1,2, \ldots \\ q^k p^{Kj}\Gamma(Kj + k)/\Gamma(Kj)k! \text{ for } j \geq 1, \text{ all } k = 0,1,2, \ldots . \end{cases}$$

One then verifies that, for each $k = 0,1,2, \ldots$, $\sum_{j=0}^{\infty} p(j)f_j(k) = q(k)$ as given, and the rule follows from D26. In fact, for $k = 0$,

$$\sum_{j=0}^{\infty} p(j)f_j(0) = p(0)f_0(0) + \sum_{j=1}^{\infty} p(j)f_j(0) = e^{-\lambda} \cdot 1 + \sum_{j=1}^{\infty} \left\{(e^{-\lambda}\lambda^j/j!) \cdot (p^{Kj})\right\} = e^{-\lambda} + \sum_{j=1}^{\infty} e^{-\lambda}(\lambda p^K)^j/j! = \sum_{j=0}^{\infty} e^{-\lambda}(\lambda p^K)^j/j! = e^{-\lambda}e^{\lambda p^K}$$

$= e^{-\lambda(1-p^K)} = q(0)$ as defined. Moreover, for each $k \geq 1$, we have

$$\sum_{j=0}^{\infty} p(j)f_j(k) = p(0)f_0(k) + \sum_{j=1}^{\infty} p(j)f_j(k) = 0 + \sum_{j=1}^{\infty}\Big\{(e^{-\lambda}\lambda^j/j!)$$

$$\cdot\ (q^k p^{Kj}\Gamma(Kj + k)/\Gamma(Kj)k!\Big\} = q(k)$$

as defined for $k = 1,2, \ldots$.

D30. $q(k) = \begin{cases} e^{-\lambda};\ k = 0 \\ \sum_{1\leq j\leq k/N} e^{-\lambda}\binom{k-1}{Nj-1}\left(\lambda p^N\right)^j q^{k-Nj}/j!;\ k = N, N+1, \ldots, \end{cases}$

$\lambda > 0$, N integer ≥ 1, $0 < p < 1$, $q = 1 - p$.

R_X. Sample $e^{-\lambda}\lambda^j/j!$ for $j \in \{0,1,2, \ldots\}$ by D2. If $j = 0$, set $k = 0$. If $j \geq 1$, sample $\binom{k-1}{Nj-1}q^{k-Nj}p^{Nj}$ for $k \in \{Nj, Nj+1, \ldots\}$ by D11, with $s = Nj$.

J. Define $p(j) = e^{-\lambda}\lambda^j/j!$; $j = 0,1,2, \ldots$, and for each such j, define the function

$$f_j(k) = \begin{cases} 1 \text{ for } j = 0,\ k = 0;\ 0 \text{ for } j = 0,\ k = 1,2,3, \ldots \\ \binom{k-1}{Nj-1}q^{k-Nj}p^{Nj} \text{ for } j \geq 1,\ k = Nj, Nj+1, \ldots, \end{cases}$$

the domain D of (j,k) being all lattice points for which $j \geq 0$, $k \geq Nj$, i.e., all lattice points $j \geq 0$, $k \geq 0$ on or above the line $k = Nj$. One can verify that, for each $k = 0,N,N+1, \ldots$, $\sum_{(j,k)\varepsilon D} p(j)f_j(k) = q(k)$ as defined above, and the rule follows from D26. (C130 is a continuous analogue.) In fact, for $k = 0$, we have $\sum_{(j,0)\varepsilon D} p(j)f_j(0) = p(0)f_0(0)$ $= e^{-\lambda}\cdot 1 = q(0)$, as defined, and for each $k = N, N+1, \ldots$,

$$\sum_{(j,k)\varepsilon D} p(j)f_j(k) = \sum_{1\leq j\leq k/N} p(j)\, f_j(k) = \sum_{1\leq j\leq k/N}\Big\{(e^{-\lambda}\lambda^j/j!)\cdot\binom{k-1}{Nj-1}$$

$$\cdot\ q^{k-Nj}p^{Nj}\Big\} = q(k) \text{ as defined for } k = N, N+1, \ldots .$$

D31. $q(k) = \begin{cases} e^{-\lambda}; \; k = 0 \\ \sum_{j=1}^{k} e^{-\lambda}\binom{k-1}{j-1}(\lambda p)^j q^{k-j}/j!; \; k = 1, 2, 3, \ldots, \; \lambda > 0, \; 0 < p < 1, \end{cases}$

$q = 1 - p$.

R_x. Sample $e^{-\lambda}\lambda^j/j!$ for $j \in \{0, 1, 2, \ldots\}$ by D2. If $j = 0$, set $k = 0$. If $j \geq 1$, sample $\binom{k-1}{j-1}q^{k-j}p^j$ for $k \in \{j, j+1, \ldots\}$ by D11 with $s = j$.

J. Case $N = 1$ of D30.

D32. $q(k) = \sum_{j=0}^{N} \binom{N}{j}(\phi j)^k (pe^{-\phi})^j q^{N-j}/k!; \; k = 0,1,2, \ldots, \; \phi > 0, \; 0 < p < 1,$

$q = 1 - p$, and (N.B.!) $(\phi j)^k \equiv 1$ for $j = k = 0$ by definition.

R_x. Set j = number of $r_1, \ldots, r_N$ such that $r_i \leq p$. If $j = 0$, set $k = 0$. If $j \geq 1$, sample $e^{-\phi j}(\phi j)^k/k!$ for $k \in \{0,1,2, \ldots\}$ by D2.

J. We write $q(k) = \sum_{j=0}^{N} \left(\binom{N}{j}q^{N-j}p^j\right) \cdot (e^{-\phi j}(\phi j)^k/k!)$ in the form of D26, where $p(j) = \binom{N}{j}q^{N-j}p^j$ is a binomial density for $j \in \{0,1, \ldots, N\}$, and for each such j, $f_j(k) = e^{-\phi j}(\phi j)^k/k!$ is a density for $k \in \{0,1,2, \ldots\}$. For $j \geq 1$, $f_j(k)$ is a Poisson density with parameter $\xi = \phi j > 0$ as in D2, whereas for $j = 0$

$$f_j(k) = \begin{cases} 1 \text{ for } k = 0 \\ 0 \text{ for } k > 0 \end{cases}$$

by definition. The rule follows from D26 and D8.

D33. $q(k) = \int_a^b dx\, f(x,k); \; k = K, K+1, \ldots, \; f(x,k)$ density for $a < x < b$,

$k = K, K + 1, \ldots$

R_x. Sample the x-marginal density

$$p(x) = \sum_{k=K}^{\infty} f(x,k)$$

for x on (a,b). For this x, sample the x-dependent discrete k-density

$p(k|x) = f(x,k)/p(x)$

for $k \geq K$.

J. This is the continuous-discrete version of D24, where explanations are given which are applicable to all four combinations of domain for the variables (j,k).

D34. $q(k) = \int_0^{\xi} x^k e^{-x}\, dx/k!\xi = \sum_{\nu=k+1}^{\infty} e^{-\xi}\xi^{\nu-1}/\nu!;\ k = 0,1,2,\ \ldots,\ \xi > 0.$

R_x. For $x = r_0\xi$, sample $e^{-x}x^k/k!$ for $k \in \{0,1,2,\ \ldots\}$ by D2.

J. The rule follows from D33. For, the density $f(x,k) = x^k e^{-x}/k!\xi$ on $0 < x < \xi$, $k = 0,1,2,\ \ldots,$ has marginal k-density $\int_0^{\xi} dx\ f(x,k) = q(k)$ as given, and marginal x-density $p(x) = \sum_{k=0}^{\infty} f(x,k) = (e^{-x}/\xi) \sum_{k=0}^{\infty} x^k/k!$ $= 1/\xi$. Moreover, the x-dependent k-density is $p(k|x) = f(x,k)/p(x)$ $= e^{-x}x^k/k!;\ k \in \{0,1,2,\ \ldots\}$, the Poisson density of D2. The value of x results from C1, since $r_0 = \int_0^{x} p(x)\ dx = x/\xi$.

Note. The identification of q(k) with the sum in D34 follows from F3A, since

$$\int_0^{\xi} x^k e^{-x}\, dx/k!\xi = \xi^{-1}\left(1 - \int_{\xi}^{\infty} x^k e^{-x}\, dx/k!\right) = \xi^{-1}\left(1 - e^{-\xi}\sum_0^k \xi^{\nu}/\nu!\right)$$

$$= \xi^{-1}e^{-\xi}\sum_{k+1}^{\infty}\xi^{\nu}/\nu! = \sum_{k+1}^{\infty} e^{-\xi}\xi^{\nu-1}/\nu!\ .$$

D35. $q(k) = \int_a^b dx\ p(x)\ f_x(k);\ k = K,\ K+1,\ \ldots,\ p(x)$ density for x on (a,b), $f_x(k)$ discrete k-density for each value of parameter x on (a,b).

R_x. Sample p(x) for x on (a,b). For this x, sample density $f_x(k)$ for $k \geq K$.

J. Corollary of D33, with $f(x,k) \equiv p(x)f_x(k)$. Note that p(x) is the marginal x density, and $f_x(k) = f(k|x)$.

D36. $q(k) = (b - a)^{-1}\left\{e^{-a}\sum_0^k a^{\nu}/\nu! - e^{-b}\sum_0^k b^{\nu}/\nu!\right\};\ k = 0,1,2,\ \ldots,$

$0 < a < b$.

R_X. Set $x = a + (b - a)r$. For this x, sample the Poisson density $e^{-x}x^k/k!$ for $k \in \{0,1,2, \ldots\}$ by D2.

J. For the uniform density $p(x) = 1/(b - a)$ on (a,b), and the Poisson x-dependent k-density $f_x(k) = e^{-x}x^k/k!$ for $k \in \{0,1,2, \ldots\}$ one has

$$\int_a^b dx\, p(x)f_x(k) = (b - a)^{-1} \int_a^b dx\, e^{-x}x^k/k! = (b - a)^{-1}$$

$$\cdot \left\{ \int_a^\infty - \int_b^\infty dx\, e^{-x}x^k/k! \right\} = (b - a)^{-1} \left\{ \sum_0^k e^{-a}a^\nu/\nu! - \sum_0^k e^{-b}b^\nu/\nu! \right\}$$

by F3A, and this is the given $q(k)$. Following D35, we may sample $p(x)$ for $x = a + (b - a)r$ on (a,b) by C11, and for this x, sample $f_x(k) = e^{-x}x^k/k!$ for $k \geq 0$ by D2.

<u>D37. $q(k) = q^k p^s \Gamma(s + k)/\Gamma(s)k!$</u>; $k = 0,1,2, \ldots$, $0 < p < 1$, $q = 1 - p$, $s > 0$, $s \in \{1,2,3, \ldots\}$. (For integral s, use D11 for $n \geq s$, and set $k = n - s \geq 0$.)

R_X. Sample $u^{s-1}e^{-u}/\Gamma(s)$ for u on $(0,\infty)$ by C64 or R27. Set $x = uq/p$. For this x, sample $e^{-x}x^k/k!$ for $k \in \{0,1,2, \ldots\}$ by D2.

J. For $p(x) = (p/q)^s x^{s-1} e^{-xp/q}/\Gamma(s)$ on $(0,\infty)$, and $f_x(k) = e^{-x}x^k/k!$, $k \in \{0,1,2, \ldots\}$, one finds $\int_0^\infty dx\, p(x)f_x(k) = ((p/q)^s/\Gamma(s)k!)$

$$\cdot \int_0^\infty dx\, x^{s+k-1}e^{-x/q} = q(k)$$ as given. By D35, we may sample $p(x)$ for $x > 0$, and for this x, sample $f_x(k)$ for $k \geq 0$. But for $x = uq/p$, one has $p(x)\, dx = u^{s-1}e^{-u}\, du/\Gamma(s)$, and the rule follows from C2.

<u>Note</u>. One can adapt Note 1 of D11 to show that $\sum_0^\infty q(k) = 1$ for non-integral s. This also follows from $\sum_0^\infty q(k) = \sum_0^\infty \int_0^\infty dx\, p(x)f_x(k)$

$$= \int_0^\infty dx\, p(x) = 1.$$

D38. $q(k) = \binom{N}{k} B(k + a, N - k + b)/B(a,b)$; $k = 0, 1, \ldots, N$, $a, b > 0$, $N \in \{1,2,3, \ldots\}$.

R_x. Sample the density $x^{a-1}(1 - x)^{b-1}/B(a,b)$ for x on $(0,1)$ by C75 or R28 if $b \neq 1$, or by C15 or C16 if $b = 1$. For this x, set k = number of $r_1, \ldots, r_N$ such that $r_i \leq x$.

J. For the density $p(x) = x^{a-1}(1 - x)^{b-1}/B(a,b)$ on $(0,1)$, and the binomial density $f_x(k) = \binom{N}{k}(1 - x)^{N-k}x^k$, $k = 0,1, \ldots, N$, with $0 < x < 1$, one finds that $\int_0^1 dx\, p(x)f_x(k) = q(k)$ as given. Following D35, we may sample $p(x)$ for x on $(0,1)$, and for this x, considered as a probability p, we may sample the binomial density $f_x(k)$ for $k \in \{0,1, \ldots, N\}$ by D8. The rule follows.

D39. $q(k) = \Gamma(s + k)B(a + s, b + k)/\Gamma(s)k!\, B(a,b)$; $k = 0,1,2, \ldots$, $a,b,s > 0$.

R_x. Sample $p(x) = x^{a-1}(1 - x)^{b-1}/B(a,b)$ for x on $(0,1)$ by C75 or R28 if $b \neq 1$, or by C15 or C16 if $b = 1$. For this x, sample $f_x(k) = (1 - x)^k x^s \Gamma(s + k)/\Gamma(s)k!$ for $k \in \{0,1,2,\ldots\}$ by D11 if s is integral, or by D37 if not.

J. For the densities defined above, one has $\int_0^1 dx\, p(x)f_x(k) = q(k)$ as given. The rule then follows from D35.

Note. Included here are the special cases: Beta-compounded geometric ($s = 1$), power-compounded negative binomial ($b = 1$), and power-compounded geometric ($s = 1 = b$). See D40, D41 for the latter.

D40. $q(k) = aB(a + 1, 1 + k) = ak!/(a + 1)(a + 2) \ldots (a + 1 + k)$; $k = 0,1,2, \ldots$, $a > 0$.

R_x1. Sample $p(x) = ax^{a-1}$ for x on $(0,1)$ by C15 or C16. For this x, sample $f_x(k) = (1 - x)^k x$ for $k \in \{0,1, \ldots\}$ by D12. (Specifically, one takes $p = x$, samples D12 for $n \geq 1$ and sets $k = n - 1$.)

J1. Case $s = 1 = b$ of D39.

R_x2. Choose any $A,B > 0$ such that $a = A/B$ (e.g., $A = a$, $B = 1$).

Sample $\hat{q}(k)$ for $k \varepsilon \{0,1,2,\ldots\}$ as in D41, R_x1.

J2. Under the substitution $a = A/B$, $x = e^{-Bu}$, one finds that

$$q(k) = \int_0^1 dx\ p(x)f_x(k) = \int_0^1 dx\ ax^{a-1}(1-x)^k x = a\int_0^1 x^a(1-x)^k\ dx$$

$$= (A/B)\int_0^\infty (e^{-Bu})^{A/B}(1-e^{-Bu})^k(Be^{-Bu}\ du) = \hat{q}(k) \text{ as in D41.}$$

<u>Note</u>. For $a = 1$, $q(k) = 1/(k+1)(k+2)$.

D41. $\hat{q}(k) = \displaystyle\int_0^\infty Ae^{-(A+B)u}(1-e^{-Bu})^k\ du;\ k = 0,1,2,\ \ldots,\ A,B > 0.$

R_x1. Sample $p(u) = Ae^{-Au}$ for $u = -A^{-1}\ \ell n\ r$ on $(0,\infty)$ by C29.

For this u, sample $f_u(k) = (1-e^{-Bu})^k e^{-Bu}$ for k on $\{0,1,2,\ \ldots\}$ by D12. (Specifically one samples D12 with $p = e^{-Bu}$ for $n \geq 1$ and sets $k = n - 1$.)

J1. One verifies that $\displaystyle\int_0^\infty p(u)f_u(k)du = \hat{q}(k)$ as above and uses D35.

R_x2. Define $a = A/B$, and sample q(k) for $k \varepsilon \{0,1,2,\ \ldots\}$ as in D40, R_x1.

J2. Under the substitution $a = A/B$, $e^{-Bu} = x$, one sees that

$$\hat{q}\ (k) = \int_0^\infty Ae^{-(A+B)u}(1-e^{-Bu})^k\ du = (A/B)\int_0^\infty \Big\{(e^{-Bu})^{A/B}(1-e^{-Bu})^k$$

$$\cdot\ (Be^{-Bu}\ du)\Big\} = a\int_0^1 x^a(1-x)^k\ dx = aB(a+1,1+k) = q(k)$$

as in D40.

<u>Note</u>. The apparent circularity is intentional, in order to justify the key word of the index.

<u>D42. $p(a/b) = (e-1)^2/(e^{a+b}-1)$</u>; a,b integers ≥ 1, $(a,b) = 1$.

R_x. Set $m = \min\left\{m;\ \sum_{q=2}^{m} \phi(q)/(e^q - 1) \geq r_1(e-1)^{-2}\right\}$, $m \geq 2$, and

$j = \min\{j;\ j \geq r_2\phi(m)\}$. List the $\phi(m)$ integers a on $\{1, 2, \ldots, m\}$ which are prime to m as $1 = a_1 < a_2 < \ldots < a_{\phi(m)}$. Set $a = a_j$, $b = m - a_j$. (For notation, see F21.)

J. Classify all a/b according to the sum $q = a + b \geq 2$. The probability of the class a/b with a particular sum q is $\sum_{a+b=q} p(a/b) = \sum_{a+b=q}\left\{(e-1)^2 \cdot (e^{a+b} - 1)^{-1}\right\} = (e-1)^2(e^q - 1)^{-1} \sum_{\substack{a,b>1\\(a,b)=1\\a+b=q}} 1 = (e-1)^2(e^q-1)^{-1}\phi(q)$,

and according to D42, all a/b belonging to a particular class are equally likely. The rule follows.

Note 1. $\sum p(a/b) = 1$ is a consequence of Liouville's identity (F21). For using the above results for the probability of a class q, we have

$$\sum p(a/b) = \sum_{q=2}^{\infty}\sum_{a+b=q} p(a/b) = (e-1)^2 \sum_{q=2}^{\infty} \phi(q)/(e^q - 1) = (e-1)^2$$

$$\cdot \sum_{q=2}^{\infty} \phi(q)e^{-q}/(1 - e^{-q}) = (e-1)^2\left\{\sum_{q=1}^{\infty} \phi(q)e^{-q}/(1-e^{-q})\right.$$

$$\left. - (e^{-1}/(1 - e^{-1}))\right\} = (e-1)^2\{e^{-1}/(1-e^{-1})^2 - e^{-1}/(1-e^{-1})\}$$

(where we have substituted $y = e^{-1}$ in F21)

$$= (e-1)^2(e^{-1}/(1-e^{-1}))\{1/(1-e^{-1}) - 1\}$$
$$= (e-1)^2\,(e^{-1}/(1-e^{-1}))^2 = (e-1)^2(1/(e-1))^2 = 1.$$

Note 2. The version (D40) in the second Sampler appears to be wrong.

D43. $\prod_{1}^{\infty} p_i(\nu_i)$. Methods are given in [18,19], based on "Poisson sequences of trials," for producing random sequences of integers with stipulated asymptotic densities.

C-INDEX

Continuous Densities

<u>Note</u>. If a Γ or B density involves an exponent not in the set {1/2, 1, 3/2, 2, ... }, see R27, 28 if not explicitly referred to.

C1. $p(\nu)$.	General continuous.
C2. $p(y)\,dy = q(x)\,dx$.	Change of variable.
C3. $\sum_j a_j(\nu)$.	Sum of positive terms, interpolated density.
C4. $1 + \nu^2$.	Thomson scattering.
C5. $q(u) = \frac{d}{du} P\{f(\nu) \leq u\}$.	Density for value of a function.
C6. $\prod_1^n p_i(\nu_i)$.	Vector density.
C7. $q(u) = \frac{d}{du} P\{f(\nu_1, \ldots, \nu_n) \leq u\}$.	Density for value of a function.
C8. $q(u) = F(u)A(u)$.	Special case of C7. A geometric device.
C9. $s(u), p(u), q(u)$.	Densities for $\nu_1 + \nu_2$, $\nu_1\nu_2$, ν_2/ν_1.
C10. $p(\nu)$.	A general device avoiding C1.
C11. $1/(b - a)$.	Uniform.
C12. $c_0 + c_1\nu$.	Linear, disk radius.
C13. $c_0 + c_1\nu + c_2\nu^2$.	Certain quadratics, shell radius.
C14. $1 - \nu^2$.	Special quadratic.
C15,C16. u^{m-1}, $m > 0$.	Power, sphere radius, completely degenerate gas momentum.
C17. $\sum_0^\infty a_j\nu^j$.	Power series, Butler.
C18. u^{-1} .	Hyperbolic.
C19. $1/(1 + \beta x)$.	Truncated type VI, Bradford. [22, v. 3; p. 89]

C20,21. v^{-m-1}, $m > 0$. — Power.

C22. $e^x/(\beta + e^x)^{m+1}$. — Generalized logistic I. [22, v. 3; p. 17]

C23. $e^{-y}/(1 + \beta^{-1}e^{-y})^{m+1}$. — Generalized logistic II. [22, v.3; p. 17]

C24. $(y + 1)^{-(m+1)}$, $m > 0$. — Pareto.

C25. $e^{Bx}/(1 + e^{Bx})^2$. — Kahn, approximate normal.

C26. $g(\hat{w}) + g(-\hat{w})$. — Folded.

C27. $2s(\hat{w})$. — Folded symmetric.

C28. $s(w)$. — Symmetric.

C29. $e^{-a\nu}$. — Exponential, Laplace I, decay time, collision distance.

C30. $e^{-y}/(1 - \lambda e^{-y})$. — Log series-compounded exponential.

C31. $e^{-a_1 u} - e^{-a_2 u}$. — Difference of exponentials.

C32. $\sum_1^n F_i^n \cdot e^{-a_i u}$. — Exponential convolute.

C33. $y^{a_1 - 1} - y^{a_2 - 1}$. — Difference of powers.

C34. $\sum_1^n F_i^n \cdot y^{a_i - 1}$. — Power convolute.

C35. $\sum_1^n B_j e^{-a_j u}$. — Sum of exponentials.

C36. $B_1 e^{-a_1 u} + B_2 e^{-a_2 u}$. — Hyperexponential, residence times.

C37. $e^{-a|u-b|}$. — Bilateral exponential, Laplace II.

C38. $x^{b-1}e^{-ax^b}$. — Weibull.

C39. $\cosh \theta$. — Hyperbolic cosine.

C40. $\sinh \theta$. — Hyperbolic sine.

C41. $(1 + \theta x) \exp\{-(x + \frac{1}{2}\theta x^2)\}$. — Linear failure rate, lifetimes. [22, v. 3; p. 268]

C42. $[1 + \theta(1 - e^{-x})] \cdot \exp\{-[x + \theta(x + e^{-x} - 1)]\}$.	Life times. [22, v.3; p. 268]
C43. $\exp(z - e^{-z})$.	Extreme value. [22, v.2; p. 277]
C44. $e^{-(z-\zeta)/\theta} \exp\{-e^{-(z-\zeta)/\theta}\}$.	2 parameter extreme value. [22, v.2; p. 277]
C45. $u^{n-1}e^{-u}$; $n = 1,2,3, \ldots$.	Γ-type, Erlangian, Pearson.
C46. $(1 + y)e^{-y/K}$.	Sum of Γ-types.
C47. $\nu^{n-1}/(e^{\nu} - 1)$, $n = 2,3, \ldots$.	Planck Type, Bose-Einstein.
C48. $u^{2n-1}/(e^{u^2} - 1)$, $n = 2,3, \ldots$.	Version of C47.
C49. $\nu^{2n-1}e^{-\nu^2}$, $n = 1,2, \ldots$.	Gauss type, Rayleigh, Maxwell.
C50. Re^{-R^2} .	Gauss type, n = 1.
C51. $e^{-\nu^2}$; $(0,\infty)$.	Error function.
C52. $u^{-1} \exp\{-(\ell n^2 u)/2b\}$.	Log normal.
C53. $\left[1 + \left(\frac{x - \xi}{\lambda}\right)^2\right] \cdot \exp\left\{-(1/2)\left[\gamma + \delta \sinh^{-1}\left(\frac{x - \xi}{\lambda}\right)\right]\right\}$.	S_U curves. [9; p. 126]
C54. $\left(\frac{x - \xi}{\lambda}\right)^{-1}\left(1 - \frac{x - \xi}{\lambda}\right)^{-1} \cdot \exp\left\{-(1/2)\left[\gamma + \delta\ \ell n \frac{x - \xi}{\xi + \lambda - x}\right]\right\}$.	S_B curves. [9; p. 130]
C55. $\exp\{-\ell n^2 u\}$.	Pseudo log-normal.
C56. $(x - \theta)^{-1} \cdot \exp\{-[\ell n(x - \theta) - \zeta]^2/2b\}$.	3-parameter log-normal, Cobb, Douglas. [22, v.2; p. 113]
C57. $\cosh(\xi w/\sigma^2) \cdot \exp\{-(w^2 + \xi^2)/2\sigma^2\}$.	Normal symmetric sum. [22, v.3; p.136]
C58. $\cosh(\xi\hat{w}/\sigma^2) \cdot \exp\{-(\hat{w}^2 + \xi^2)/2\sigma^2\}$.	Folded normal. [22, v.3; p. 136]
C59. e^{-x^2}; $(-\infty,\infty)$.	Normal version.

C60. $e^{-y^2/2}$; $(-\infty,\infty)$.	Normal.
C61. $u^{2n-1}e^{-u^2}$, $n = 1/2, 3/2, \ldots$.	Gauss type, Maxwell speed.
C62. $\nu^{2n-1}/(e^{\nu^2} - 1)$, $n = 3/2, 5/2, \ldots$.	Planck version.
C63. $u^{n-1}/(e^{u} - 1)$, $n = 3/2, 5/2, \ldots$.	Planck type.
C64. $\nu^{n-1}e^{-\nu}$, $n = 1/2, 3/2, \ldots$.	Γ-type, Maxwell energy, fission spectrum.
C65. $y^{A-1}\ell n^{n-1}(1/y)$.	Power-log power.
C66. $t^{np-1}e^{-t^p}$	Γ-version.
C67. $e^{-\phi t/\sigma}\exp\{-\rho e^{-t/\sigma}\}$.	Gompertz. [22, v.3; p. 271]
C68. $(1 + (x/a))^{ab}e^{-bx}$.	Transition type III. [9; p. 78]
C69. $1/x^{n+1}e^{b/x}$.	Transition type V. [9; p. 81]
C70. $x^{-(n+1)}e^{-a^2/2x}$.	One sided stable, recurrence times.
C71. $x \operatorname{csch} x$	$x \cdot$ hyperbolic cosecant.
C72. $u^{n-1}E_N(u)$.	Schlömilch, neutron diffusion.
C73. $u^{n-1}K_N(u)$.	Bessel.
C74. $\nu^{n-1}e^{-\nu}/(1 - \Lambda^2 e^{-2\nu})$; $(0,\infty)$.	Lemma for R21.
C75. $v^{m-1}(1 - v)^{n-1}$, $z^{m-1}/(1 + z)^{m+n}$, $\sin^{2m-1}\theta \cos^{2n-1}\theta$, $w^{mp-1}(1 - w^p)^{n-1}$.	Beta types, powers of sin, cos. (See R28.)
C76. $(x - a)^{m-1}(b - x)^{n-1}$.	Pearson types I, II, general Beta. [22, v.3; p. 37]
C77. $(x - b)^R/(x - a)^Q$.	Pearson type VI. [22, v.2; p.13; v.3; p. 87]
C78. $1/(e^{x/2} + e^{-x/2})^{2m}$.	Logistic power, power of sech-square. [22, v.3; pp. 5,17]
C79. $e^{-mx/\sigma}/(1 + \rho e^{-x/\sigma})^{m+n}$.	4-parameter generalized logistic. [22, v.3; p. 271]

C80. $e^{-mx/\sigma}(1 - \rho e^{-x/\sigma})^{n-1}$. — 4-parameter generalized exponential. [22, v.3; p. 271]

C81. $(1 - (x/a)^2)^{n-1}$. — Transition type II. [9; p. 74]

C82. $z^{m-1}/(1 + z),\ 0 < m < 1$. — Restricted Beta.

C83. $x(x - a)^{m-1}(b - x)^{n-1}$. — $x \cdot$ Beta.

C84. $x^{m-1}(1 - x)^{n-1}/(x + a)^{m+n}$. — Modified Beta.

C85. $(a + x)^{m-1}(a - x)^{n-1}$. — Centered Beta.

C86. $F(x) + x^{-2}F(x^{-1})$. — Reflected density.

C87. $(x^{m-1} + x^{n-1})/(1 + x)^{m+n}$. — Reflected Beta.

C88. $1\Big/\left[1 + \left|\frac{x - \theta}{\lambda}\right|^{1/m}\right]^{m+n}$. — Generalized Cauchy, Rider. [22, v.2; p. 162]

C89. $\exp\left\{-\sum_{1}^{N} \nu_i^2\right\}$. — N-normal, Maxwell velocity.

C90. $p(\Omega),\ \Omega = (\omega_1, \ldots, \omega_N)$. — Uniform (isotropic) direction in N-space, point on unit N-sphere $|\Omega| = 1$.

C91. $F\left(\left(\sum_{1}^{N} \nu_i^2\right)^{1/2}\right)$. — Radially symmetric density.

C92. $s^{(N/2)-1}e^{-s/2b}$. — χ^2 density.

C93. $u^{N-1}e^{-u^2/2b}$. — χ density.

C94. $\mu^{(N/2)-1}e^{-N\mu/2b}$. — Mean square, χ^2/N.

C95. $\rho^{N-1}e^{-N\rho^2/2b}$. — Root mean square, $(\chi^2/N)^{1/2}$.

C96. $1/(1+(t^2/N))^{(N+1)/2}$. — Student's t.

C97. $1/(c^2 + (\zeta - \zeta_0)^2)^m$. — Pearson type VII. [22, v.2; p. 13; v.3; p. 114]

C98. $1/(1 + t^2)$. — Cauchy.

C99. $1/[1 + ((z - \theta)/\lambda)^2]$. — 2-parameter Cauchy. [22, v.2; p. 154]

C100.	$[1 + ((w^2 + \theta^2)/\lambda^2)] \cdot [1 + 2((w^2 + \theta^2)/\lambda^2) + ((w^2 - \theta^2)/\lambda^2)^2]^{-1}$.	Cauchy symmetric sum, $(-\infty,\infty)$ [22, v.2; p. 163]
C101.	$2 \cdot$ (C100).	Folded Cauchy, $(0,\infty)$. [22, v.2; p. 163]
C102.	sech x .	Hyperbolic secant. [1; p. 64]
C103.	$1/(1 - u^2)^{1/2}$.	Sine of uniform angle.
C104.	$F^{(M/2)-1}/(1 + (MF/N))^{(M+N)/2}$.	Snedecor's F.
C105.	$E^{M-1}/(1 + (ME^2/N))^{(M+N)/2}$.	Square root of Snedecor's F, rms/rms.
C106.	$y^{n-1}e^{-\xi y}$; $(1,\infty)$.	Lemma for R23.
C107.	$z^{n-1}e^{-\xi z/\eta}$; (η,∞) .	Residual Γ-density.
C108.	ze^{-z}; (η,∞) .	Residual Γ-density ($n = 2$), Carey-Drijard.
C109.	$\nu^{n-1}e^{-a\nu}/(1 - \Lambda^2 e^{-2a\nu})$; $(1,\infty)$.	Lemma for R24.
C110.	$\begin{cases} u; & (0,1) \\ 2 - u; & (1,2) \end{cases}$.	Triangular, sum of two random numbers.
C111.	$\begin{cases} 4(x - a)/(c - a)^2 \\ 4(c - x)/(c - a)^2 \end{cases}$.	Symmetric triangular, tine. [22, v.3; p. 64]
C112.	$1 - \lvert x \rvert$.	Centered triangular. [22, v.3; p. 64]
C113.	$\begin{cases} a_1(x) \\ a_2(x) \end{cases}$.	Composite.
C114.	$\begin{cases} h(x - a)/(b - a) \\ h(c - x)/(c - b) \end{cases}$.	General triangular.
C115.	$\begin{cases} e^{ax} \\ e^{-bx} \end{cases}$	Asymmetric Laplace. [22, v.3; p. 31]
C116.	$a_i(x)$; (x_i, x_{i+1}) .	General composite.
C117.	$\begin{cases} px/a^2 \\ pq^{i-1}\{(1 + ip)a - px\}/a^2 \end{cases}$.	Binomial-uniform, traffic flow. [22, v.3; p. 70]
C118.	$1/(e^x + b + e^{-x})$, $-2 < b < 2$.	Symmetric exponential I. [22, v.3; p. 15]
C119.	$1/(e^x + e^{-x})$.	Hyperbolic secant. [22, v.3; p. 15]

C120. $1/(e^{x} + 2 + e^{-x})$. Logistic, sech-square, growth curve, symmetric exponential II. [22, v.2; p. 244; v.3; p.3]

C121. $1/(e^{x} + b + e^{-x})$, $b > 2$. Symmetric exponential III. [22, v.3; p. 15]

C122. $1/(b + 2 \cosh \alpha(y - y_o))$. Champernowne, income, Perks. [22, v.2; p. 242]

C123. $1/t\{t/t_o)^{\alpha} + b + (t/t_o)^{-\alpha}\}$. Champernowne, income. [22, v.2; p. 243]

C124. $\int_a^b dx\ f(x,y)$. Marginal, composition, Butler.

C125. $\left(e^{-ay^2} - e^{-by^2}\right)/y^2$. Marginal normal.

C126. $\int_0^\infty dx\ x^N e^{-x^2/2}$ Non-central t. [22, v.3; p. 204]

$\cdot \exp\{-((xy/N^{1/2}) - \delta)^2/2\}$.

C127. $\int_0^\infty dx\ x^{(n-4)/2} e^{-x/2H^2}$ Sample covariance. [22, v.3; p. 231]

$\cdot \exp\{-(y - (\rho Kx/H))^2/2K^2(1 - \rho^2)x\}$.

C128. $(e^{-ay} - e^{-by})/y$. Exponential marginal.

C129. $\left(e^{-ay^{1/n}} - e^{-by^{1/n}}\right)/y^{1/n}$. Exponential marginal, $n \neq 1$.

C130. $\int_a^y dx\ f(x,y)$. Triangular marginal, composition.

C131. $(1/2)(g(w) + g(-w))$. Symmetric sum.

C132. $\Phi(w - \rho)e^{-\rho w} + \Phi(-w - \rho)e^{\rho w}$. Compound Laplace. [22, v.3; p. 32]

C133. $\Phi\left(\frac{u - \zeta}{\sigma} - \frac{\sigma}{\phi}\right)e^{-\frac{u-\zeta}{\phi}} + \Phi\left(-\frac{u - \zeta}{\sigma} - \frac{\sigma}{\phi}\right)e^{\frac{u-\zeta}{\phi}}$. 3-parameter compound Laplace. [22, v.3; p.32]

C134. $e^{-y} \int_0^y dx\, x^{n-1}/(y - x)^n$. Marginal Gamma.

C135. $\int_{y}^{b} dx\, f(x,y)$. Marginal, triangular region.

C136. $\int_y^{y_b} dx\, t(x)/t_1$. Tail-end density.

C137. $\int_y^\infty dx\, x^{n-1} e^{-Bx}$. General Gamma tail-end.

C138. $e^{-By} \sum_0^{n-1} (By)^\nu/\nu!$. Gamma tail-end, n integral.

C139. $b^m - y^m$. Power tail-end.

C140. $\int_a^b dx\, p(x) f_x(y)$. Marginal, composition, Butler.

C141. $\int_0^1 dx\, x^{m-(3/2)} \exp(-y^2/2bx)$. Romanowski, modulated normal, equinormal (m = 1), radico-normal (m = 3/2), lineo-normal (m = 2). [22, v.3; p. 276]

C142. $\{(1 + ay)e^{-ay} - (1 + by)e^{-by}\}/y^2$. Time between calls, uniform-compounded exponential. [2; p. 69]

C143. $P^{k-1}(x)p(x)[1 - P(x)]^{N-k}$. General order statistics.

C144. A. $p(x)[1 - P(x)]^{N-1}$, Min, max, median statistics. [22, v.2; p.3]

B. $P^{N-1}(x)p(x)$,

C. $p(x)[P(x)(1 - P(x))]^M$.

C145. $(x - a)^{k-1}(b - x)^{N-k}$. Order statistics (uniform). [22, v.3; p. 38]

C146. $x^{k-1}(1 - x)^{N-k}$. Order statistics (random numbers). [22, v.3; p. 38]

C147. $e^{-x} \exp(-k\, e^{-x}) \cdot [1 - \exp(-e^{-x})]^{N-k}$. Order statistics (extreme value). [22, v.2; p. 279]

C148. $e^{-(N-k+1)x}/(1 + e^{-x})^{N+1}$.

Order statistics (logistic). [22, v.3; p.8]

C149. $\left[1/2 + (1/\pi) \arctan\left(\frac{x - \theta}{\lambda}\right)\right]^{k-1}$

$\cdot \left[1/2 - (1/\pi) \arctan\left(\frac{x - \theta}{\lambda}\right)\right]^{N-k}$

$\cdot \left[1 + \left(\frac{x - \theta}{\lambda}\right)^2\right]^{-1}$.

Order statistics (Caunchy). [22, v.2; p. 157]

C150. $x^{b-1} e^{-ax^b (N-k+1)}$

$\cdot \left[1 - e^{-ax^b}\right]^{k-1}$.

Order statistics (Weibull). [22, v.2; p. 254]

C151. $(1 - e^{-x})^{k-1} e^{-(N-k+1)x}$.

Order statistics (exponential). [22, v.2; p. 214]

C152. $[\Gamma_x(n)]^{k-1} (x^{n-1} e^{-x}/\Gamma(n))$

$\cdot [1 - \Gamma_x(n)]^{N-k}$.

Order statistics (Gamma). [22, v.2; p. 191]

C153. $(m/x)(\beta/x)^{m(N-k+1)}$

$\cdot [(i - (\beta/x)^m]^{k-1}$.

Order statistics (Pareto). [22, v.2; p. 241]

C154. $t^{-3/2} \exp\{-\lambda(t - \mu)^2/2\mu^2 t\}$.

Inverse Gaussian, first passage time. [22, v.2; p. 138]

C155. $t^{-3/2} \exp\{-(d - vt)^2/2\beta t\}$.

Brownian motion with drift. [22, v.2; p. 138]

C156. $x^{m-1} y^{n-1} F(x,y)$.

Bivariate with marginal Beta.

C157. $x^{m-1} y^{n-1}/(1 - x - y)^n$.

Bivariate with Beta marginals.

C158. $\exp\{-Q/2(1 - \rho^2)\}$,

$$Q = \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1 - \mu_1}{\sigma_1}\right) \cdot \left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2 .$$

General 2-variable normal.

C159. $\exp\{-Q/2(1 - \rho^2)\}$,

$$Q = x_1^2 - 2\rho x_1 x_2 + x_2^2 .$$

Centered reduced 2-variable normal.

C160. $\exp\left\{-\sum x_i a_{ij} x_j\right\}$. n-variable normal.

C161. $P(\alpha,\alpha')$ Klein-Nishina total cross-section. See R16, 17, 30 for polarized case.

Continuous Densities

C1. $p(\nu)$; (a,b).

R_x. Define $P(\nu) = \int_a^\nu p(\nu)\, d\nu$, $P_1(\nu) = \int_\nu^b p(\nu)\, d\nu$. Set $\nu = P^{-1}(r_0)$ or

$\nu = P_1^{-1}(r_1)$.

J. $\int_a^\nu p(\nu)\, d\nu = r$ is equivalent to $p(\nu)\, d\nu = dr$, i.e., the probability of ν on $(\nu, \nu + d\nu)$ is the probability of the corresponding random number r on $(r, r + dr)$. This is the "fundamental principle" of sampling, usually inapplicable, since solution of $P(\nu) = r$ for ν is seldom easy.

Note. Since $P(\nu) + P_1(\nu) \equiv 1$, $r_0 = P(\nu)$ is equivalent to $P_1(\nu) = 1 - r_0 = r_1$.

C2. $p(y)\, dy = q(x)(\pm\, dx)$; $y = f(x)$ monotone.

R_x. If preferable, sample $q(x)$ for x, set $y = f(x)$.

J. The probability of y on $(y, y + dy)$ is that for the corresponding x on $(x, x + dx)$.

Note 1. C1 is the special case $q(x) \equiv 1$ on $(0,1)$.

Note 2. Observe that $\{p(f(x)) \cdot |dy/dx|\}\, dx = \{q(x)\}\, dx$.

A similar rule applies to the n-variable case, with $|dy/dx|$ replaced by the absolute Jacobian $|\det[\partial_{y_i}/\partial_{x_j}]| \equiv J$ of the transformation

$$\begin{aligned} y_1 &= f_1(x_1, \ldots, x_n) \\ &\vdots \\ y_n &= f_n(x_1, \ldots, x_n) \ . \end{aligned}$$

Thus $\{p(f_1(x), \ldots, f_n(x)) \cdot J\}\, dx_1 \ldots dx_n = \{q(x_1, \ldots, x_n)\}\, dx_1 \ldots dx_n$.

C3. $p(\nu) = \sum_1^J a_j(\nu)$; (a,b), $a_j(\nu) \geq 0$.

R_x. Define $A_j = \int_a^b a_j(\nu)\, d\nu$. Set $K = \min\left\{k;\ \sum_1^k A_j \geq r_0\right\}$. Sample density

$a_K(\nu)/A_K$ for ν on (a,b).

J. Under the rule, A_j is the probability of sampling the j-th density, so the total chance of ν on $(\nu, \nu + d\nu)$ is

$$\sum_1^J A_j(a_j(\nu)/A_j)\ d\nu = \sum_1^J a_j(\nu)\ d\nu = p(\nu)\ d\nu, \text{ as required.}$$

Note that $\sum_1^J A_j = \sum_1^J \int_a^b a_j(\nu)\ d\nu = \int_a^b p(\nu)\ d\nu = 1.$

Note 1. The rule provides an elegant way of sampling an interpolated density $p(\nu) = \alpha_1 p_1(\nu) + \alpha_2 p_2(\nu)$, $p_1(\nu)$, $p_2(\nu)$ densities on (a,b), $\alpha_1, \alpha_2 > 0$, $\alpha_1 + \alpha_2 = 1$. (Lee Carter.)

Note 2. If we set $f(j,\nu) = a_j(\nu)$, we recognize $p(\nu)$ as the ν-marginal density in the discrete-continuous case of D24, and the above rule is that of D24 adapted to this case.

C4. $p(\nu) = (3/8)(1 + \nu^2)$; $(-1,1)$.

R_x1. If $r_0 \leq 3/4$, set $\nu = 2r_1 - 1$. Otherwise set $\nu = (2r_1 - 1)^{1/3}$.

J1. The rule is an obvious consequence of C3. We write

$$p(\nu) = a_1(\nu) + a_2(\nu), \text{ where}$$

$$a_1(\nu) = 3/8 \qquad\qquad a_2(\nu) = (3/8)\nu^2$$

$$A_1 = \int_{-1}^{1} a_1(\nu)\ d\nu = 3/4 \qquad\qquad A_2 = \int_{-1}^{1} a_2(\nu)\ d\nu = 1/4$$

$$a_1(\nu)/A_1 = 1/2 \qquad\qquad a_2(\nu)/A_2 = 3/2\ \nu^2,$$

and use C1 to set

$$r_1 = \int_{-1}^{\nu} (1/2)\ d\nu \equiv (1/2)(\nu + 1) \text{ or } r_1 = \int_{-1}^{\nu} (3/2)\nu^2\ d\nu = (1/2)(\nu^3 + 1)$$

obtaining

$$\nu = 2r_1 - 1 \qquad \text{or} \qquad \nu = (2r_1 - 1)^{1/3}.$$

R_x2. Generate r_1, r_2. If $r_2 \leq (3/4)(1 + (1/3)r_1^2)$, set $\alpha = r_1$. Otherwise set $\alpha = (4r_2 - 3)^{1/2}$. Set $\nu = \pm\ \alpha$ with probability 1/2.

J2. This is an application of C13 to the density

$$\overline{p}(\alpha) = (3/4)(1 + \alpha^2);\ (0,1),$$

and of C28 for choice of sign. We find from C13 that $f(r) = (3/4) \cdot (1 + (1/3)r^2)$, with $f'(r) = r/2$, $f'(0) = 0$, $\lambda(s) = (4s - 3)^{1/2}$. The rule therefore follows from C13, part (a).

C5. $q(u) = \frac{d}{du} P\{f(\nu) \leq u\} = \frac{d}{du} \int_{\{f(\nu) \leq u\}} p(\nu)\, d\nu$; (c,d), $p(\nu)$ density for ν on (a,b), $f(\nu)$ function on (a,b), $c = \min f(\nu)$, $d = \max f(\nu)$.

R_x. Sample $p(\nu)$ for ν on (a,b). Set $u = f(\nu)$.

J. $q(u)$ is the density for the value u of the function $f(\nu)$, since $P\{f(\nu) \leq u\}$ is its distribution function, and $q(u) = dP/du$.

Note. The idea here and in related densities is that if the given density $q(u)$ can be recognized as of the form

$$\frac{d}{du} \int_{\{f(\nu) \leq u\}} p(\nu)\, d\nu$$

for some function $f(\nu)$ and density $p(\nu)$, then $q(u)$ may be sampled in the way described.

C6. $p(\nu) = \prod_1^n p_i(\nu_i)$; $p_i(\nu_i)$ densities on various domains.

R_x. Sample each $p_i(\nu_i)$ for ν_i; set vector $\nu = (\nu_1, \ldots, \nu_n)$.

J. $p(\nu)$ is the probability density for the vector $\nu = (\nu_1, \ldots, \nu_n)$ where the components ν_i are independent.

C7. $q(u) = \frac{d}{du} P\{f(\nu_1, \ldots, \nu_n) \leq u\} = \frac{d}{du} \int_{\{f(\nu) \leq u\}} \prod_1^n p_i(\nu_i)$; (c,d), $p_i(\nu_i)$ densities on (a,b), $f(\nu)$ function on $(a,b) \times \ldots \times (a,b)$, $c = \min f(\nu)$, $d = \max f(\nu)$.

R_x. Sample each $p_i(\nu_i)$ for ν_i on (a,b). Set $u = f(\nu_1, \ldots, \nu_n)$.

J. $q(u)$ is the density for the value u of the function $f(\nu_1, \ldots, \nu_n)$ under the density $p_1(\nu_1) \ldots p_n(\nu_n)$, since $P\{f(\nu) \leq u\}$ is its distribution function, and $q(u) = dP/du$.

Note 1. As in C5, the idea is that if a given density can be recognized as of the form

$$\frac{d}{du} \int_{\{f(\nu)\leq u\}} \prod_{1}^{n} p_i(\nu_i)\, d\nu_i$$

for some function $f(\nu_1, \ldots, \nu_n)$ and densities $p_1(\nu_1), \ldots, p_n(\nu_n)$, then $q(u)$ can be sampled as in the rule. Examples are C8, C9.

Note 2. The obvious extension to the case of a density $p(\nu_1, \ldots, \nu_n)$ of non-independent variables is left to the reader.

C8. $q(u) = F(u)A(u)$; $(0,\infty)$, $A(u) = dV/du$, where

$$V(u) = \int_{\{f(\nu_1,\ldots,\nu_n)\leq u\}} \prod_{1}^{n} d\nu_i$$ for some function $f(\nu_1, \ldots, \nu_n)$, and

$$F(f(\nu_1, \ldots, \nu_n)) = \prod_{1}^{n} p_i(\nu_i),$$ a product of densities on $(0,\infty)$.

Rx. Sample each $p_i(\nu_i)$ for ν_i on $(0,\infty)$; set $u = f(\nu_1, \ldots, \nu_n)$.

J. Such a $q(u)$ is the density for the value u of the function $f(\nu_1, \ldots, \nu_n)$ under the density $\prod_{1}^{n} p_i(\nu_i)$, since

$$\frac{d}{du} \int_{\{f(\nu)\leq u\}} \prod_{1}^{n} p_i(\nu_i)\, d\nu_i = \frac{d}{du} \int_{\{f(\nu)\leq u\}} F(f(\nu_1, \ldots, \nu_n)) \prod_{1}^{n} d\nu_i$$

$$= \frac{d}{du} \int_{0}^{u} F(u)A(u)\, du = F(u)A(u) = q(u)$$

as defined. The rule then follows from C7.

Note 1. This highly artificial looking device is the key to sampling many important densities. In the sequel, the factor $A(u)$ is identified as one of the areas in F7, F8. See C45 for a first example.

Note 2. The argument in (J) is based on the consideration of the one-parameter family of surfaces $f(\nu_1, \ldots, \nu_n) = u$. See [26; p. 323].

Note 3. The extension to the case of a density $p(\nu_1, \ldots, \nu_n)$ of non-independent variables is left to the reader.

C9. $s(u) = \int_0^u p_1(\nu_1)p_2(u - \nu_1)\, d\nu_1$; $(0,\infty)$, p_1,p_2 densities on $(0,\infty)$.

$p(u) = \int_0^a p_1(\nu_1)\nu_1^{-1}p_2(u\nu_1^{-1})\, d\nu_1$; $(0,\infty)$, p_1 density on $(0,a)$, a finite or infinite, p_2 density on $(0,\infty)$.

$q(u) = \int_0^\infty p_1(\nu_1)\nu_1 p_2(u\nu_1)\, d\nu_1$; $(0,\infty)$, p_1,p_2 densities on $(0,\infty)$.

$q_1(u) = \int_0^\infty p_1(\nu_1)\nu_1 p_2(u\nu_1)\, d\nu_1$; $(-\infty,\infty)$, p_1 density on $(0,\infty)$, p_2 density on $(-\infty,\infty)$.

$q_2(u) = 2\int_0^\infty p_1(\nu_1)\nu_1 p_2(u\nu_1)\, d\nu_1$; $(-\infty,\infty)$, p_1,p_2 even densities on $(-\infty,\infty)$.

R_x. In all cases, sample $p_1(\nu_1)$ for ν_1, $p_2(\nu_2)$ for ν_2. For $s(u)$, set $u = \nu_1 + \nu_2$. For $p(u)$, set $u = \nu_1\nu_2$. For $q(u)$, $q_1(u)$, or $q_2(u)$, set $u = \nu_2/\nu_1$.

J. For each of the functions $f(\nu_1,\nu_2) = \nu_1 + \nu_2$, $\nu_1\nu_2$, and ν_2/ν_1, one verifies that $\frac{d}{du}\int_{\{f(\nu)\leq u\}} p_1(\nu_1)p_2(\nu_2)\, d\nu_1\, d\nu_2$ has the form of the corresponding density above. Verification of $s(u)$ is given in F2A, and of $q_2(u)$ in F2B. The rule follows from C7.

C10. $p(\nu)$; (a,b). A general device.

R_x. Define $P(\nu) = \int_a^\nu p(\nu)\, d\nu$, $P_1(\nu) = \int_\nu^b p(\nu)\, d\nu$ as in C1, and $f(r) = r^{-1}P(a + (b - a)r)$, $g(r) = r^{-1}P_1(b - (b - a)r)$, $0 < r < 1$.

a. If $f(r)$, in particular if $p(\nu)$, is increasing, set

$$u = \begin{cases} r_1 & \text{if } r_2 \leq f(r_1) \\ f^{-1}(r_2) & \text{if } r_2 > f(r_1), \end{cases}$$ and $\nu = a + (b - a)u$.

b. If $g(r)$ is increasing, in particular if $p(\nu)$ is decreasing, set

$$u = \begin{cases} r_1 \text{ if } r_2 \leq g(r_1) \\ g^{-1}(r_2) \text{ if } r_2 > g(r_1), \end{cases} \text{ and } \nu = b - (b - a)u.$$

<u>Note 1</u>. The functions $f(r)$ and $g(r)$ are well-defined for r on $(0,1)$, and $\lim_{r\to 0} f(r) = (b - a)p(a)$, $\lim_{r\to 0} g(r) = (b - a)p(b)$. Since both functions are are increasing by assumption, and $f(1) = 1 = g(1)$, both have values on $(0,1)$.

If $r_2 > f(r_1)$, then $f^{-1}(r_2)$ is well-defined on $(0,1)$; in fact, there exists a u on $(r_1,1)$ such that $u = f^{-1}(r_2)$, i.e., $r_2 = f(u)$. A similar remark applies to $g(r)$ if $r_2 > g(r_1)$. The rule is therefore well-defined. In effect, it sets $u = \max\{r_1, f^{-1}(r_2)\}$ in (a) and $u = \max\{r_1, g^{-1}(r_2)\}$ in (b), where u is on $(0,1)$, and hence ν is on (a,b).

<u>Note 2</u>. If $p(\nu)$ is increasing, then $f(r)$ is necessarily increasing. For, by F1, we have

$$f'(r) = r^{-2}\{r(b - a)p(a + (b - a)r) - P(a + (b - a)r)\} > 0,$$

since, for $p(\nu)$ increasing, it is clear geometrically that

$$r(b - a)p(a + (b - a)r) > \int_a^{a+(b-a)r} p(\nu)\, d\nu.$$

Similarly $p(\nu)$ decreasing implies $g(r)$ increasing.

J. We justify the rule in case (a) for $f(r)$ increasing. An analogous argument applies to case (b) for $g(r)$ increasing.

First note that for $\nu = a + (b - a)u$, with u on $(0,1)$, one has $p(\nu)\, d\nu = p(a + (b - a)u)(b - a)\, du \equiv q(u)\, du$, so by C2, we may sample the density $q(u) = (b - a)p(a + (b - a)u)$ for u on $(0,1)$, and set $\nu = a + (b - a)u$. We shall prove that this $q(u)$ is in fact the density for the value u of the function

$$F(r,s) = \begin{cases} r \text{ if } s \leq f(r) \\ f^{-1}(s) \text{ if } s > f(r), \end{cases} \quad r,s \in (0,1),$$

under the uniform density $p_1(r)p_2(s)$ on the unit square, where $p_1(r) \equiv 1 \equiv p_2(s)$ on $(0,1)$. By C7, we may therefore set $r = r_1$, $s = r_2$ and $u = F(r_1,r_2)$, which gives the rule in case (a). Now it is clear geometrically that

$$P\{F(r,s) \le u\} = \int_{\{F(r,s)\le u\}} dr\, ds = uf(u) = P(a + (b - a)u),$$

by definition of f(r). Hence

$\frac{d}{du} P\{F(r,s) \le u\} = \frac{d}{du} P(a + (b - a)u) = (b - a)p(a + (b - a)u) = q(u)$ as defined above, so q(u) is indeed the density for the value u of the function F(r,s).

Note 3. The above device is only practical if f,g are more easily invertible than P, P_1, which is indeed the case for all linear densities $p(\nu)$, and for certain quadratic densities (Cf. C12, C13, C14). For further details, see [7].

C11. $p(\nu) = 1/(b - a)$; (a,b).

R_x. Set $\nu = a + (b - a)r_0$.

J. By C1, we set $r_0 = \int_a^\nu d\nu/(b - a) = (\nu - a)/(b - a)$ and solve for ν.

C12. $p(\nu) = C^{-1}(c_0 + c_1\nu)$; (a,b), $c_1 \ne 0$, $C = (b - a)[c_0 + (c_1/2)(b + a)]$.

R_x. If $c_1 > 0$, set

$\nu = a + \max\{(b - a)r_1, (b + a + 2c_0c_1^{-1})r_2 - 2(a + c_0c_1^{-1})\}$.

If $c_1 < 0$, set

$\nu = b - \max\{(b - a)r_1, - (b + a + 2c_0c_1^{-1})r_2 + 2(b + c_0c_1^{-1})\}$.

J. The rule is an immediate consequence of C10.

Note. For the radius ν of a uniform disk, with density $p(\nu) = 2\nu/(b^2 - a^2)$ on (a,b), the rule sets $\nu = a + \max\{(b - a)r_1, (b + a)r_2 - 2a\}$, as compared with C1, which would set $\nu = \{a^2 + (b^2 - a^2)r_0\}^{1/2}$. For a = 0, b = 1, $p(\nu) = 2\nu$, the comparison is $\nu = \max\{r_1, r_2\}$ against $\nu = r_0^{1/2}$.

C13. $p(\nu) = C^{-1}h(\nu)$, $h(\nu) = c_0 + c_1\nu + c_2\nu^2$, (a,b), $c_2 \ne 0$,

$C = (b - a)\{c_0 + (c_1/2)(b + a) + (c_2/3)(b^2 + ba + a^2)\}$, for certain cases.

R_x. Define $f(r) = C^{-1}(b - a)\{h(a) + (1/2)h'(a)(b - a)r + (c_2/3)(b - a)^2r^2\}$,

with $f'(r) = C^{-1}(b - a)^2\{(1/2)h'(a) + (2/3)c_2(b - a)r\}$,

$f'(0) = C^{-1}(b - a)^2h'(a)/2$,

$$f'(1) = C^{-1}(b - a)^2\{(1/2)h'(a) + (2/3)c_2(b - a)\}$$
$$= \frac{2}{3} C^{-1}(b - a)^2 c_2\{b + (a/2) + (3c_1/4c_2)\}, \text{ and}$$
$$g(r) = C^{-1}(b - a)\{h(b) - (1/2)h'(b)(b - a)r + (c_2/3)(b - a)^2 r^2\}, \text{ with}$$
$$g'(r) = C^{-1}(b - a)^2\{-(1/2)h'(b) + (2/3)c_2(b - a)r\},$$
$$g'(0) = -C^{-1}(b - a)^2 h'(b)/2,$$
$$g'(1) = C^{-1}(b - a)^2\{-(1/2)h'(b) + (2/3)c_2(b - a)\}$$
$$= -(2/3)C^{-1}(b - a)^2 c_2\{a + (b/2) + (3c_1/4c_2)\}.$$

a. If $c_2 > 0$ and $f'(0) \geq 0$, or if $c_2 < 0$ and $f'(0) > 0$, $f'(1) \geq 0$,

$$\text{set } \nu = \begin{cases} a + (b - a)r_1 & \text{if } r_2 \leq f(r_1) \\ a + \lambda(r_2) & \text{if } r_2 > f(r_1), \end{cases}$$

where $\lambda(s) = (1/2)\left\{-(3h'(a)/2c_2) + \operatorname{sgn} c_2\left[(3h'(a)/2c_2)^2 + \frac{12}{c_2}\left(\frac{Cs}{b - a} - h(a)\right)\right]^{1/2}\right\}$.

b. If $c_2 > 0$ and $g'(0) \geq 0$, or if $c_2 < 0$ and $g'(0) > 0$, $g'(1) \geq 0$,

$$\text{set } \nu = \begin{cases} b - (b - a)r_1 & \text{if } r_2 \leq g(r_1) \\ b - \mu(r_2) & \text{if } r_2 > g(r_1), \end{cases}$$

where $\mu(s) = (1/2)\left\{(3h'(b)/2c_2) + \operatorname{sgn} c_2\left[(3h'(b)/2c_2)^2 + \frac{12}{c_2}\left(\frac{Cs}{b - a} - h(b)\right)\right]^{1/2}\right\}$.

J. The functions $f(r), g(r)$, computed as in C10, both represent parabolas, each opening up if $c_2 > 0$, and down if $c_2 < 0$.

a. If $c_2 > 0$, $f(r)$ opens up, and $f'(0) \geq 0$ insures that $f(r)$ is increasing on $(0,1)$. If $c_2 < 0$, $f(r)$ opens down, and $f'(0) > 0$, $f'(1) \geq 0$ again insures $f(r)$ increasing on $(0,1)$. Hence we may apply the rule of C10 as in case (a). Inversion of $s = f(r)$ gives $(b - a)r = \lambda(s)$ as defined. Note that sgn c_2 governs the choice of sign in the solution of the corresponding quadratic, since this determines the type of concavity of $f(r)$.

b. An identical argument shows $g(r)$ increasing under the stated conditions, so we may follow the rule of C10 as given in case (b).

<u>Note 1.</u> If $c_2 < 0$, the conditions in (a), (b) are

a. $a < -(c_1/2c_2)$, $b \leq -(a/2) - (3c_1/4c_2)$.

b. $b > -(c_1/2c_2)$, $a \geq -(b/2) - (3c_1/4c_2)$.

No parabola can satisfy <u>both</u> conditions. For, assuming (a) and (b), we have $b \leq -(a/2) - (3c_1/4c_2) \leq (b/4) + (3c_1/8c_2) - (3c_1/4c_2) = (b/4) - (3c_1/8c_2)$.

Hence $(3/4)b \leq -(3c_1/8c_2)$ or $b \leq -(c_1/2c_2)$ in conflict with the first part of (b).

Note 2. The method, when applicable, is indicated if inversion in C1 involves a difficult cubic. (See C14.) For the radius ν of a uniform spherical shell, $p(\nu) = 3\nu^2/(b^3 - a^3)$, C1 sets

$\nu = \{a^3 + (b^3 - a^3)r_o\}^{1/3}$ whereas C13 sets

$$\nu = \begin{cases} a + (b - a)r_1 \text{ if } r_2 \leq f(r_1) \\ a + \lambda(r_2) \text{ if } r_2 > f(r_1), \text{ where} \end{cases}$$

$f(r) = \{3a^2 + 3a(b - a)r + (b - a)^2r^2\}/(b^2 + ab + a^2) \equiv \hat{A} + r(\hat{B} + r\hat{C})$,

$\lambda(s) = (1/2)\{-3a + [4(b^2 + ba + a^2)s - 3a^2]^{1/2}\} \equiv \hat{D} + [\hat{E}s + \hat{F}]^{1/2}$,

$\hat{A}, \ldots, \hat{F}$ stored.

C14. <u>$p(\nu) = (3/2)(1 - \nu^2)$</u>; (0,1).

R_x. Generate r_1, r_2. If $r_2 \leq r_1(3 - r_1)/2$, set $\nu = 1 - r_1$.

Otherwise set $\nu = (1/2)[-1 + (9 - 8r_2)^{1/2}]$.

J. The rule is an application of C13, part (b). For, one finds that $g(r) = (3/2)(r - (1/3)r^2)$, with $g'(r) = 3/2(1 - (2/3)r)$, $g'(0) > 0$, $g'(1) > 0$, and $\mu(s) = (1/2)[3 - (9 - 8s)^{1/2}]$.

C15. <u>$q(u) = mb^{-m}u^{m-1}$</u>; (0,b), $m = k/\ell$, $k,\ell \in \{1,2,3, \ldots\}$.

R_x. Set $u = b(\max\{r_1, \ldots, r_k\})^\ell$.

J. For $u = bv^\ell$, one has $q(u)\,du = kv^{k-1}\,dv$, so by C2, one may sample the density kv^{k-1} for v on (0,1), and set $u = bv^\ell$. But for the uniform densities $p_1(\nu_1) = \ldots = p_k(\nu_k) \equiv 1$, ν_i on (0,1), one sees that

$$\frac{d}{d\nu} P(\max\{\nu_1, \ldots, \nu_k\} \leq v) = \frac{d}{dv} \int_{\max\{\nu_1,\ldots,\nu_k\} \leq v} d\nu_1 \ldots d\nu_k = \frac{d}{dv}(v^k)$$

$= kv^{k-1}$, so that kv^{k-1} is the density for the function $f(\nu_1, \ldots, \nu_k) = \max\{\nu_1, \ldots, \nu_k\}$ under the density $p_1(\nu_1) \ldots p_k(\nu_k)$, and the rule follows from C7. (see C146, Note 2.)

<u>Note</u>. kv^{k-1} is the density for the function $\max\{r_1, \ldots, r_k\}$ of k random numbers. Cf. C12, Note, for $k = 2$, $\ell = 1$, $m = 2$.

C16. $q(u) = C^{-1}u^{m-1}$; (a,b), $0 \leq a < b$, $m > 0$, $C = (b^m - a^m)/m$.

R_x. Set $u = \{a^m + (b^m - a^m)r_0\}^{1/m}$.

J. Using C1, we set $r_0 = \int_a^u q(u)\,du = (u^m - a^m)/(b^m - a^m)$, and solve for u.

C17. $p(\nu) = \sum_0^\infty a_j \nu^j$; $(0,1)$, $a_j > 0$.

R_x. Define $A_j = \int_0^1 a_j \nu^j \, d\nu = a_j/(j + 1)$.

Set $K = \min\left\{k; \sum_0^k A_j \geq r_0\right\}$. Set $\nu = r_1^{1/(K+1)}$ or set ν $= \max\{r_1, \ldots, r_{K+1}\}$.

J. The rule follows from C3, and from C16 or C15, since we may write $p(\nu) = \sum_0^\infty A_j(a_j(\nu)/A_j)$, set K as in the rule, and then sample the density $a_K(\nu)/A_K = (K + 1)\nu^K$ for ν on $(0,1)$.

C18. $q(u) = C^{-1}u^{-1}$; (a,b), $0 < a < b$, $C = \ln(b/a)$.

R_x. Set $u = ae^{Cr_o} = a^{1-r_o}b^{r_o}$.

J. Using C1, we set $r_0 = \int_a^u C^{-1}\,du/u = C^{-1}\ln u/a$, and solve for u.

C19. $p(x) = \beta/(1 + \beta x)\ln(1 + \beta)$; $(0,1)$, $\beta > -1$.

R_x. Set $x = \beta^{-1}\{-1 + \exp[r_o \ln(1 + \beta)]\}$.

J. The rule follows from C1, where we set

$$r_0 = \int_0^x p(x)\,dx = \ln(1 + \beta x)/\ln(1 + \beta), \text{ and solve for } x.$$

Note. For $x = \beta^{-1}(-1 + u)$, one has $p(x)\,dx = du/u\ln(1 + \beta) = q(u)\,du$ on $(1,1 + \beta)$ as in C18. This, of course, results in the same rule.

C20. $p(\nu) = m\beta^m \nu^{-m-1}$; (β,∞), $\beta > 0$, $m = k/\ell$; $k, \ell \in \{1,2,3, \ldots\}$.

R_x. Set $\nu = \beta/(\max\{r_1, \ldots, r_k\})^\ell$.

J. For $v = 1/u$, one finds that $p(v)\,dv = m(1/\beta)^{-m}u^{m-1}(-du)$ on $(0,1/\beta)$, with $m = k/\ell$ as in C15. Hence by C15 we set $u = (1/\beta)(\max\{r_1, \ldots, r_k\})^{\ell}$ and $v = 1/u$ by C2.

C21. $p(v) = C^{-1}v^{-m-1}$; (β,α), $0 < \beta < \alpha \leq \infty$, $m > 0$, $C = (\beta^{-m} - \alpha^{-m})/m$.

R_x. Set $v = 1/\{\beta^{-m} - (\beta^{-m} - \alpha^{-m})r_0\}^{1/m}$, or set $v = 1/\{\alpha^{-m} + (\beta^{-m} - \alpha^{-m})r_1\}^{1/m}$.

J. By C1, we set $r_0 = \int_{\beta}^{v} C^{-1}v^{-m-1}\,dv = (\beta^{-m} - v^{-m})/(\beta^{-m} - \alpha^{-m})$, or r_1

$$= \int_{v}^{\alpha} C^{-1}v^{-m-1}\,dv = (v^{-m} - \alpha^{-m})/(\beta^{-m} - \alpha^{-m}),$$ and solve

for v.

Note. For $\alpha = \infty$, we have the simple rule:
set $v = \beta/(1 - r_0)^{1/m}$, or $v = \beta/r_1^{1/m}$.

C22. $q(x) = m\beta^m e^x/(\beta + e^x)^{m+1}$; $(-\infty,\infty)$, $\beta,m > 0$.

R_x. Sample $p(v) = m\beta^m v^{-m-1}$ for v on (β,∞) by C20 or C21. Set $x = \ell n(v - \beta)$.

J. The function $x = \ell n(v - \beta)$ increases from $x = -\infty$ to $x = \infty$ for v on (β,∞), and $q(x)\,dx = m\beta^m v^{-m-1}\,dv$ on (β,∞). The rule follows from C2.

Note. For $m = \beta = 1$, $q(x) = e^x/(1 + e^x)^2 = 1/(e^x + 2 + e^{-x})$ as in C120.

C23. $r(y) = m\beta^{-1}e^{-y}/(1 + \beta^{-1}e^{-y})^{m+1}$; $(-\infty,\infty)$, $\beta,m > 0$.

R_x. Sample $p(v) = m\beta^m v^{-m-1}$ for v on (β,∞) by C20 or C21. Set $y = -\ell n(v - \beta)$.

J. For $y = -x$, one has
$r(y)\,dy = m\beta^{-1}e^x(-dx)/(1 + \beta^{-1}e^x)^{m+1} = m\beta^m e^x(-dx)/(\beta + e^x)^{m+1}$ on $(-\infty,\infty)$, as in C22. Thus the rule follows from C22 and C2.

C24. $q(y) = m(y + 1)^{-(m-1)}$; $(0,\infty)$, $m > 0$.

R_x. Set $y = v - 1$, where v is obtained from C20 or C21, with $\beta = 1$, $\alpha = \infty$.

J. For $y = v - 1$, one has $q(y)\,dy = mv^{-m-1}\,dv$ on $(1,\infty)$.

C25. $f(x) = Be^{Bx}/(1 + e^{Bx})^2$; $(-\infty,\infty)$, $B > 0$.

R_x. Set $x = B^{-1}\ell n(r_1^{-1} - 1)$.

J. Under the transformation $y = 1 + e^{Bx}$, which increases from $y = 1$ to $y = \infty$ for x on $(-\infty,\infty)$, one has $f(x)\,dx = dy/y^2$, with y on $(1,\infty)$. Hence by C2, we may sample $1/y^2$ for y on $(1,\infty)$, and set $x = B^{-1}\ell n(y - 1)$. But from C1, setting $r_1 = \int_y^\infty dy/y^2$ gives $y = r_1^{-1}$, and the rule follows. This is of course also the result of C20 or C21.

C26. $h(\hat{w}) = g(\hat{w}) + g(-\hat{w})$; $(0,\infty)$, $g(z)$ density on $(-\infty,\infty)$.

R_x. Sample density $g(z)$ for z on $(-\infty,\infty)$. Set $\hat{w} = |z|$.

J. Obvious.

C27. $h(\hat{w}) = 2s(\hat{w})$; $(0,\infty)$, $s(w)$ symmetric density on $(-\infty,\infty)$.

R_x. Sample $s(w)$ for w on $(-\infty,\infty)$. Set $\hat{w} = |w|$.

J. Special case of C26.

C28. $s(w)$; $(-\infty,\infty)$, $s(w)$ symmetric: $s(-w) = s(w)$.

R_x. Sample density $2s(\hat{w})$ for $\hat{w}$ on $(0,\infty)$. Set $w = \pm \hat{w}$ with probability 1/2.

J. Obvious.

Note. C26, 27, 28, 131 are closely related, and each has its uses independently.

C29. $p(\nu) = ae^{-a\nu}$; $(0,\infty)$, $a > 0$.

R_x. Set $\nu = -a^{-1}\ell n\, r_1$.

J. By C1, we set $r_1 = \int_\nu^\infty a\, e^{-a\nu}\, d\nu = e^{-a\nu}$, and solve for ν.

Note. This is the first link in the following chain of densities on $(0,\infty)$.

C29. $p(\nu) = e^{-\nu}$, $\nu = -\ell n r_1$, by C1.

C45. $q(u) = u^{n-1}e^{-u}/\Gamma(n)$, $n = 1,2, \ldots$, $u = -\ell n \prod_1^n r_i$, by C29, C8.

C49. $p(\nu) = 2\nu^{2n-1}e^{-\nu^2}/\Gamma(n)$, $n = 1,2, \ldots$, (2n even), $\nu = (-\ell n \prod_1^n r_i)^{1/2}$, by C45, C2.

C50. $p(R) = 2Re^{-R^2}$, $R = (-\ell n r_1)^{1/2}$, by C49, $n = 1$.

C51. $p(\nu_1) = 2e^{-\nu_1^2}/\pi^{1/2}$, $\nu_1 = R \cos\theta$, by C50, C2.

C61. $q(u) = 2u^{2n-1}e^{-u^2}/\Gamma(n)$, $n = 1/2, 3/2, \ldots$, (2n odd),
$u = \{-\ell n(r_1 \ldots r_h) + \tau^2\}^{1/2}$, by C51, C8.

C64. $p(\nu) = \nu^{n-1}e^{-\nu}/\Gamma(n)$, $n = 1/2, 3/2, \ldots,$

$u = \{-\ell n(r_1 \ldots r_h) + \tau^2\}$, by C61, C2.

C30. $q(y) = \lambda e^{-y}/(1 - \lambda e^{-y})L(\lambda)$; $(0,\infty)$, $0 < \lambda < 1$,

$L(\lambda) = -\ell n(1 - \lambda)$.

R_x1. Set $y = -\ell n\ \lambda^{-1}\{1 - \exp[-r_1 L(\lambda)]\}$.

J1. By C1, we set $r_1 = \int_y^\infty q(y)\, dy = -(1/L(\lambda))\ell n(1 - \lambda e^{-y})$,

and solve for y.

R_x2. Set $K = \min\{k; \sum_1^k \lambda^j/j \geq r_0 L(\lambda)\}$. Set $y = -K^{-1}\ell n\ r_1$.

J2. The rule results from C3. For, we may write

$q(y) = \sum_1^\infty (\lambda^j/jL(\lambda))(je^{-jy})$, set K as in the rule, and sample Ke^{-Ky} for

y on $(0,\infty)$, which gives $y = -K^{-1}\ \ell n r_1$ by C29.

C31. $q(u) = \dfrac{a_1 a_2}{a_2 - a_1}\left(e^{-a_1 u} - e^{-a_2 u}\right)$; $(0,\infty)$, $0 < a_1 < a_2$.

R_x. Set $u = -a_1^{-1}\ \ell n r_1 - a_2^{-1}\ \ell n r_2$.

J. q(u) is the density for the sum $u = \nu_1 + \nu_2$ under the density $p_1(\nu_1) \cdot p_2(\nu_2)$, where $p_i(\nu_i) = a_i e^{-a_i \nu}$ on $(0,\infty)$, $i = 1, 2$, so the rule follows from C7 and C29. In fact,

$$\frac{d}{du}\int_{\{\nu_1+\nu_2 \leq u\}} p_1(\nu_1)\, d\nu_1\, p_2(\nu_2)\, d\nu_2 = \frac{d}{du}\int_0^u p_2(\nu_2)\, d\nu_2 \int_0^{u-\nu_2} p_1(\nu_1)\, d\nu_1$$

$$\equiv \frac{d}{du}\int_0^u F(u,\nu_2)\, d\nu_2$$

$$= 1 \cdot F(u,u) - 0 \cdot F(u,0)$$

$$+ \int_0^u \frac{\partial}{\partial u}\{F(u,\nu_2)\}\, d\nu_2 \qquad \text{(cf. F1)}$$

$$= 0 - 0 + \int_0^u \left\{ p_2(\nu_2)\, d\nu_2 \frac{d}{du} \int_0^{u-\nu_2} p_1(\nu_1)\, d\nu_1 \right\}$$

$$= \int_0^u p_2(\nu_2)\, d\nu_2\, p_1(u - \nu_2)$$

$$= a_1 a_2 \int_0^u e^{-a_2\nu_2} e^{-a_1(u-\nu_2)} d\nu_2$$

$$= a_1 a_2 e^{-a_1 u} \int_0^u e^{-(a_2-a_1)\nu_2} d\nu_2$$

$$= \frac{a_1 a_2}{a_2 - a_1} e^{-a_1 u}\left(1 - e^{-(a_2-a_1)u}\right)$$

$$= \frac{a_1 a_2}{a_2 - a_1}\left(e^{-a_1 u} - e^{-a_2 u}\right) = q(u)\ du,$$

as given.

Note. One may of course use C9, but the above proof is more in conformity with the induction in C32.

C32. $q_n(u) = \sum_{i=1}^{n} F_i^n e^{-a_i u}$; $(0,\infty)$, where

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$$F_i^n = \frac{a_1 \ldots a_n}{(a_1-a_i) \ldots (a_{i-1}-a_i)(a_{i+1}-a_i) \ldots (a_n-a_i)},$$

a_i distinct > 0, $n \geq 2$.

R_x. Set $u = -\sum_1^n a_i^{-1} \ell n r_i$.

J. It can be shown by induction on $n \geq 2$ that $q_n(u)$ is the density for the sum $u = \nu_1 + \ldots + \nu_n$ under the density $p_1(\nu_1) \ldots p_n(\nu_n)$, where $p_i(\nu_i) = a_i e^{-a_i \nu_i}$, so that the rule follows from C7 and C29. The basis ($n = 2$) for the induction is provided by C31. Letting $\nu = \nu_1 + \ldots + \nu_n$, we compute

$$\frac{d}{du} \int_{\{\nu+\nu_{n+1} \leq u\}} p_{n+1}(\nu_{n+1})\, d\nu_{n+1} \prod_1^n p_i(\nu_i)\, d\nu_i$$

$$= \frac{d}{du} \int_0^u p_{n+1}(\nu_{n+1})\, d\nu_{n+1} \int_{\{\nu \leq u-\nu_{n+1}\}} \prod_1^n p_i(\nu_i)\, d\nu_i$$

$$\equiv \frac{d}{du} \int_0^u F(u,\nu_{n+1})\, d\nu_{n+1} = 1 \cdot F(u,u) - 0 \cdot F(u,0)$$

$$+ \int_0^u \frac{\partial}{\partial u} F(u,\nu_{n+1})\, d\nu_{n+1} = 0 - 0 \qquad \text{(See F1)}$$

$$+ \int_0^u p_{n+1}(\nu_{n+1})\, d\nu_{n+1} \frac{d}{du} \int_{\{\nu_1 + \ldots + \nu_n \leq u-\nu_{n+1}\}} \prod_1^n p_i(\nu_i)\, d\nu_i$$

$$= \int_0^u p_{n+1}(\nu_{n+1})\, d\nu_{n+1} q_n(u - \nu_{n+1}),$$

by the induction hypothesis.

Hence, the density for the sum $u = \nu_1 + \ldots \nu_{n+1}$ is

$$\int_0^u a_{n+1} e^{-a_{n+1}\nu_{n+1}}\, d\nu_{n+1} \sum_1^n F_i^n e^{-a_i(u-\nu_{n+1})}$$

$$= \sum_1^n F_i^n a_{n+1} e^{-a_i u} \int_0^u e^{-(a_{n+1}-a_i)\nu_{n+1}}\, d\nu_{n+1}$$

$$= \sum_1^n \frac{F_i^n a_{n+1}}{(a_{n+1}-a_i)} e^{-a_i u}\left(1 - e^{-(a_{n+1}-a_i)u}\right)$$

$$= \sum_{1}^{n} \frac{F_i^n a_{n+1}}{(a_{n+1} - a_i)} e^{-a_i u} - \sum_{1}^{n} \frac{F_i^n a_{n+1}}{(a_{n+1} - a_i)} e^{-a_{n+1} u}$$

$$= \sum_{1}^{n} F_i^{n+1} e^{-a_i u} + F_{n+1}^{n+1} e^{-a_{n+1} u} - \left(\sum_{1}^{n} F_i^{n+1} + F_{n+1}^{n+1} \right) e^{-a_{n+1} u}$$

$$= \sum_{1}^{n+1} F_i^{n+1} e^{-a_i u} - 0 = q_{n+1}(u). \qquad \text{(See F10A.)}$$

<u>Note 1.</u> $\int_0^\infty q_n(u)\, du = \int_0^\infty \sum_1^n F_i^n e^{-a_i u}\, du = \sum_1^n (F_i^n / a_i) = 1$ by F10B.

<u>Note 2.</u> As the derivative of an increasing distribution function, it appears that $q_n(u)$ is non-negative on $(0,\infty)$. This is an interesting inequality for which we have no direct proof. Consider the case $n = 3$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$.

<u>Note 3.</u> It is shown in C45 that $u^{n-1} e^{-u}/\Gamma(n)$ is the density for the sum $u = \nu_1 + \ldots + \nu_n$ under the density $p_1(\nu_1) \ldots p_n(\nu_n)$, where $p_i(\nu_i) = a_i e^{-a_i \nu_i}$, and all $a_i = 1$.

C33. $p(y) = \dfrac{a_1 a_2}{a_2 - a_1}\left(y^{a_1 - 1} - y^{a_2 - 1}\right)$; $(0,1)$, $0 < a_1 < a_2$.

R_x. Set $y = \exp\left\{a_1^{-1} \ell n r_1 + a_2^{-1} \ell n r_2\right\}$.

J. For $y = e^{-u}$, which decreases from $y = 1$ to $y = 0$ for u on $(0,\infty)$, one has $p(y)\, dy = \dfrac{a_1 a_2}{a_2 - a_1}\left(e^{-a_1 u} - e^{-a_2 u}\right)(-du)$ as in C31. From this and C2, the rule follows.

C34. $p(y) = \sum_1^n F_i^n y^{a_i - 1}$; $(0,1)$, where the F_i^n are defined in C32.

R_x. Set $y = \exp\left\{\sum_1^n a_i^{-1} \ell n r_i\right\}$.

J. For $y = e^{-u}$, one has $p(y)\, dy = q_n(u)(-du)$, for the $q_n(u)$ in C32. The rule follows from this and C2.

C35. $p(u) = \sum_{1}^{n} B_j e^{-a_j u}$; $(0,\infty)$, $B_j > 0$.

R_x. Set $K = \min\left\{k; \sum_{1}^{k} (B_j/a_j) \geq r_0\right\}$, and $u = -a_K^{-1} \ln r_1$.

J. The rule follows from C3 and C29, since

$$A_j = \int_0^\infty B_j e^{-a_j u}\, du = (B_j/a_j), \text{ and } a_j(u)/A_j = a_j e^{-a_j u}.$$

C36. $p(u) = (2a^2/\tau)e^{-2au/\tau} + 2((1-a)^2/\tau)e^{-2(1-a)u/\tau}$; $(0,\infty)$, $\tau > 0$, $0 < a \leq 1/2$.

R_x. If $r_o \leq a$, set $u = -(\tau/2a)\ln r_1$. Otherwise set $u = -(\tau/2(1-a))\ln r_1$.

J. Special case of C35, with $B_1 = 2a^2/\tau$, $a_1 = 2a/\tau$, $B_1/a_1 = a$, $B_2 = 2((1-a)^2/\tau)$, $a_2 = 2(1-a)/\tau$, $B_2/a_2 = 1-a$.

C37. $q(u) = (a/2)e^{-a|u-b|}$; $(-\infty,\infty)$, $a > 0$, b arbitrary.

R_x. Set $\hat{w} = -a^{-1}\ln r_o$, $w = \pm \hat{w}$ with probability 1/2, and $u = b + w$.

J. For $u = b + w$, one has $q(u)\,du = (a/2)e^{-a|w|}\,dw = s(w)\,dw$, where $s(w)$ is a symmetric density on $(-\infty,\infty)$. Hence we may sample $s(w)$ and set $u = b + w$. But by C28, we may sample $2s(\hat{w}) = ae^{-a\hat{w}}$ for $\hat{w}$ on $(0,\infty)$, and set $w = \pm \hat{w}$ with probability 1/2. By C29, we set $\hat{w} = -a^{-1}\ln r_o$, and the rule follows.

C38. $p(x) = abx^{b-1}e^{-ax^b}$; $(0,\infty)$, $a,b > 0$.

R_x. Set $x = (-a^{-1}\ln r_o)^{1/b}$.

J. For $x = \nu^{1/b}$, one has $p(x)\,dx = ae^{-a\nu}\,d\nu$ on $(0,\infty)$ as in C29, where we set $\nu = -a^{-1}\ln r_o$. The rule follows from C2.

C39. $p(\theta) = S^{-1}\cosh\theta$; $(0,t)$, $S = \sinh t$.

R_x1. Define $A_1 = (e^t - 1)/2S$. If $r_0 \leq A_1$, set $\theta = \ln[1 + r_1(e^t - 1)]$. Otherwise set $\theta = -\ln[1 - r_1(1 - e^{-t})]$.

J1. Following C3, we write $p(\theta) = a_1(\theta) + a_2(\theta)$, where

$$a_1(\theta) = e^{\theta}/2S, \qquad a_2(\theta) = e^{-\theta}/2S,$$

$$A_1 = (e^t - 1)/2S, \qquad A_2 = (1 - e^{-t})/2S,$$

$$a_1(\theta)/A_1 = e^{\theta}/(e^t - 1). \qquad a_2(\theta)/A_2 = e^{-\theta}/(1 - e^{-t}).$$

By C1, we set $r_1 = \int_0^{\theta} e^{\theta}\, d\theta/(e^t - 1)$, or $r_1 = \int_0^{\theta} e^{-\theta}\, d\theta/(1 - e^{-t})$, and solve for θ, obtaining the values of θ in the rule.

R_x2. Set $\theta = \ln\{Sr_0 + [(Sr_0)^2 + 1]^{1/2}\}$.

J2. By C1, we may set $r_0 = \int_0^{\theta} p(\theta)\, d\theta = (e^{\theta} - e^{-\theta})/2S$. Solving for θ gives the setting of the rule, where the choice of sign is obviously mandatory.

<u>C40. $p(\theta) = C^{-1} \sinh \theta$;</u> $(0,t)$, $C = (\cosh t) - 1$.

R_x. Set $\theta = \ln\{(Cr_0 + 1) + [(Cr_0 + 1)^2 - 1]^{1/2}\}$.

J. By C1, one may set $r_0 = \int_0^{\theta} p(\theta)\, d\theta = (e^{\theta} + e^{-\theta} - 2)/2C$, obtaining $\theta = \ln\{(Cr_0 + 1) \pm [(Cr_0 + 1)^2 - 1]^{1/2}\}$. For either sign, $r_0 = 0$ gives $\theta = 0$. The choice of (+) sign is indicated, since $r_0 = 1$ then gives $\theta = \ln\{\cosh t + [\cosh^2 t - 1]^{1/2}\} = \ln\{\cosh t + \sinh t\} = \ln e^t = t$. One may also note that, for $r_0 > 0$, $(Cr_0 + 1) - [(Cr_0 + 1)^2 - 1]^{1/2} < 1$, i.e., $(Cr_0)^2 < (Cr_0)^2 + 2(Cr_0)$, since $C = (1/2)(e^t + e^{-t}) - 1 > 0$ for $t > 0$, whereas $e^{\theta} \geq 1$.

<u>C41. $p(x) = (1 + \theta x) \exp\{-(x + (1/2)\theta x^2)\}$;</u> $(0,\infty)$, $\theta > 0$.

R_x. Set $x = \theta^{-1}\{-1 + [1 - 2\theta \ln r]^{1/2}\}$.

J. For the increasing function $\nu = x + (1/2)\theta x^2$, we see that $p(x)\,dx = e^{-\nu}\, d\nu$. By C29, we sample $e^{-\nu}$ for $\nu = -\ln r$ on $(0,\infty)$, and set $x = x(\nu)$. But solving the quadratic $\nu = x + (1/2)\theta x^2$ for x gives $x = x(\nu) = \theta^{-1}\{-1 \pm [1 + 2\theta\nu]^{1/2}\}$, where the (+) sign is obviously required.

<u>C42. $q(x) = [1 + \theta(1 - e^{-x})] \exp\{-[x + \theta(x + e^{-x} - 1)]\}$;</u> $(0,\infty)$, $\theta > 0$.

R_x. Set $\nu_0 = -\ln r_0$. Solve the equation $\nu_0 = x + \theta(x + e^{-x} - 1)$ for $x = x(\nu_0)$ on $(0,\infty)$. See Note below.

J. For x on $(0,\infty)$ the function $\nu = x + \theta(x + e^{-x} - 1)$ is increasing from $\nu = 0$ to $\nu = \infty$, with $d\nu = [1 + \theta(1 - e^{-x})]\,dx$. Hence $q(x)\,dx = e^{-\nu}\,d\nu$ as in C29. The rule follows from C2.

<u>Note</u>. The equation $\nu_0 = x + \theta(x + e^{-x} - 1)$ may be solved for x by Newton's method. For the function

$$f(x) = x + \theta(x + e^{-x} - 1) - \nu_0$$

one has $f(0) = -\nu_0 < 0$, and $f(\nu_0) = \theta(\nu_0 + e^{-\nu_0} - 1) > 0$, since $e^{-\nu_0} > 1 - \nu_0$. Moreover,

$$f'(x) = 1 + \theta(1 - e^{-x}) > 0$$

and $f''(x) = \theta e^{-x} > 0$ on $0,\infty$.

Since the curve $y = f(x)$ is increasing and concave up, Newton's sequence $x_0 = \nu_0$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{\nu_0 + \theta\left[1 - (1 + x_n)e^{-x_n}\right]}{1 + \theta\left[1 - e^{-x_n}\right]}$$

converges to x from above.

<u>C43. $q(y) = \exp(-y - e^{-y})$;</u> $(-\infty,\infty)$.

R_x. Set $y = -\ln(-\ln r)$.

J. For $y = -\ln\nu$, one has $q(y)\,dy = e^{-\nu}(-d\nu)$ on $(0,\infty)$ as in C29. From this and C2, the rule follows.

<u>Note</u>. This is a special case $\theta = 1$, $\zeta = 0$ of C44.

<u>C44. $p(z) = \theta^{-1}e^{-\frac{z-\zeta}{\theta}}\exp\left\{-e^{-\frac{z-\zeta}{\theta}}\right\}$;</u> $(-\infty,\infty)$, $\theta > 0$, ζ arbitrary.

R_x. Set $z = \zeta - \theta\ln(-\ln r)$.

J. For $z = \zeta + \theta y$, one has $p(z)\,dz = e^{-y-e^{-y}}\,dy$ on $(-\infty,\infty)$ as in C43. The rule then follows from C2.

<u>C45. $q(u) = u^{n-1}e^{-u}/\Gamma(n)$;</u> $(0,\infty)$, $n \in \{1,2,3, \ldots\}$.

R_x. Set $u = -\ln \prod_1^n r_i$.

J. The rule is an application of C8. In fact, $q(u) = F(u)A(u)$, where $F(u) = e^{-u}$ and $A(u) = u^{n-1}/(n - 1)!$ By F7, $A(u) = dV/du$, for

$$V = \int_{\left\{ \sum_1^n \nu_i \le u,\ \nu_i > 0 \right\}} \prod_1^n d\nu_i,$$ and moreover, for the function $f(\nu) = \sum_1^n \nu_i$, one

has $F(f(\nu)) = e^{-\sum_1^n \nu_i} = \prod_1^n e^{-\nu_i}$, a product of identical densities on

$(0,\infty)$. We therefore sample each $e^{-\nu_i}$ for ν_i ($= -\ell n r_i$, by C29), and set

$$u = f(\nu) = \sum_1^n \nu_i = \sum_1^n (-\ell n r_i) = -\ell n \prod_1^n r_i.$$

Note. $q(u) = u^{n-1}e^{-u}/\Gamma(n)$ is the density for $u = \nu_1 + \ldots + \nu_n$ under the density $e^{-\nu_1} \ldots e^{-\nu_n}$, and hence $Q(u) = \int_0^u q(u)\,du = \int_0^u u^{n-1}e^{-u}\,du/(n-1)!$

is its distribution function:

$$\int_{\left\{ \sum_1^n \nu_i \le u,\ \nu_i > 0 \right\}} \prod_1^n e^{-\nu_i}\, d\nu_i.$$ $Q(u)$ may be evaluated by F3A.

C46. $m(y) = (1 + y)e^{-y/K}/(K + K^2)$; $(0,\infty)$, $K > 0$.

R_x. If $r_0 \le 1/(1 + K)$, set $y = -K\ell n r_1$. Otherwise, set $y = -K\ell n r_1 r_2$.

J. Following C3, we write $m(y) = a_1(y) + a_2(y)$, where

$a_1(y) = e^{-y/K}/(K + K^2)$,	$a_2(y) = ye^{-y/K}/(K + K^2)$,
$A_1 = 1/(1 + K)$,	$A_2 = K/(1 + K)$,
$a_1(y)/A_1 = e^{-y/K}/K$.	$a_2(y)/A_2 = ye^{-y/K}/K^2$.
For $y = Ku$,	For $y = Ku$,
$a_1(y)\,dy/A_1 = e^{-u}\,du$.	$a_2(y)\,dy/A_2 = ue^{-u}\,du$.

The rule then follows from C3, C2, C29, C45.

<u>Note</u>. To sample $\hat{m}(y) = C^{-1}(1 + y)e^{-y/K}$ for y on (0,Y), where Y >> 1, one may follow the above rule, accepting y only when $y \le Y$. In such a method, the probability of rejection is

$$(1/(K+1)) \int_Y^\infty e^{-y/K}\, dy/K + (K/(K+1)) \int_Y^\infty y e^{-y/K}\, dy/K^2 = (1/(K+1))e^{-Y/K}$$

$$+ (K/(K+1))(1 + Y/K)e^{-Y/K}$$

$$= e^{-Y/K}(1 + (Y/(1+K)).$$

<u>C47. $p(\nu) = \nu^{n-1}/(e^\nu - 1)\zeta(n)\Gamma(n)$</u>; $(0,\infty)$, $n \in \{2,3,4, \ldots\}$. (See F9D.)

R_x. Set $K = \min\left\{k;\ \sum_1^k (1/j^n) \ge r_0\zeta(n)\right\}$, and $\nu = -K^{-1}\ell n \prod_1^n r_i$.

J. We may write $p(\nu) = \zeta^{-1}(n)\Gamma^{-1}(n)\nu^{n-1}e^{-\nu}/(1 - e^{-\nu}) = \zeta^{-1}(n)\Gamma^{-1}(n)$

$\sum_1^\infty \nu^{n-1}e^{-j\nu} = \sum_1^\infty (1/j^n\zeta(n))(j^n\nu^{n-1}e^{-j\nu}/\Gamma(n))$, which is a sum of the

form in C3. We may therefore set K as in the rule, and sample the density $K^n\nu^{n-1}e^{-K\nu}/\Gamma(n)$ for ν on $(0,\infty)$. But for $\nu = u/K$, one has $K^n\nu^{n-1}e^{-K\nu}\, d\nu/\Gamma(n) = u^{n-1}e^{-u}\, du/\Gamma(n)$ as in C45. Hence we set

$u = -\ell n \prod_1^n r_i$, and $\nu = K^{-1}u$ by C2. (Noted for n = 4 by C. Barnett, E. Canfield.)

<u>C48. $q(u) = 2u^{2n-1}\Big/\left(e^{u^2} - 1\right)\zeta(n)\Gamma(n)$</u>; $(0,\infty)$, $n \in \{2,3,4, \ldots\}$.

R_x. Sample $p(\nu)$ for ν on $(0,\infty)$ as in C47. Set $u = \nu^{1/2}$.

J. For $u = \nu^{1/2}$, one has $q(u)\, du = p(\nu)\, d\nu$, and the rule follows from C2.

<u>C49. $p(\nu) = 2\nu^{2n-1}e^{-\nu^2}/\Gamma(n)$</u>; $(0,\infty)$, $n \in \{1,2, \ldots\}$, 2n even.

R_x. Set $\nu = \left(-\ell n \prod_1^n r_i\right)^{1/2}$.

J. For $\nu = u^{1/2}$, one has $p(\nu)\, d\nu = u^{n-1}e^{-u}\, du\, \Gamma(n) = q(u)$ as in C45. The rule then follows from C2.

C50. $p(R) = 2Re^{-R^2}$; $(0,\infty)$.

R_x. Set $R = (-\ell n r_1)^{1/2}$.

J. Case n = 1 of C49.

C51. $p(\nu_1) = 2e^{-\nu_1^2}/\pi^{1/2}$; $(0,\infty)$.

R_x1. Set $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, where $R = (-\ell n r_1)^{1/2}$, $\theta = (\pi/2)r'$. (Two independent samples ν_1 and ν_2 are obtained.)

J1. Under the polar transformation $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, with Jacobian R, one has

$$(2/\pi^{1/2})e^{-\nu_1^2}\, d\nu_1 \cdot (2/\pi^{1/2})e^{-\nu_2^2}\, d\nu_2 = 2Re^{-R^2}\, dR \cdot (2/\pi)\, d\theta.$$

The rule follows from C2, C50.

R_x2. Obtain $S = \hat{r}_1^2 + \hat{r}_2^2 \leq 1$ as in R1. Set $\nu_1 = \{(-\ell n r_1)/S\}^{1/2}\hat{r}_1$, $\nu_2 = \{(-\ell n r_1)/S\}^{1/2}\hat{r}_2$.

J2. The rule follows as in J1, with $\cos\theta = \hat{r}_1/S^{1/2}$, $\sin\theta = \hat{r}_2/S^{1/2}$, obtained as in R1.

R_x3. Obtain $S = \hat{r}_1^2 + \hat{r}_2^2 \leq 1$ as in R1. Set $\nu_1 = \{(-\ell n S)/S\}^{1/2}\hat{r}_1$, $\nu_2 = \{(-\ell n S)/S\}^{1/2}\hat{r}_2$.

J3. Under the transformation $(\nu_1,\nu_2) \rightarrow (\rho,\theta)$: $\nu_1 = R\cos\theta$, $\nu_2 = R\sin\theta$, with $R = (-2\ell n\rho)^{1/2}$, $(0,\infty) \times (0,\infty) \rightarrow (0,1) \times (0,\pi/2)$, with Jacobian $RR' = R(1/2)(1/R)(-2/\rho) = -1/\rho$, one has $(2/\pi^{1/2})e^{-\nu_1^2}\, d\nu_1 \cdot (2/\pi^{1/2})e^{-\nu_2^2}\, d\nu_2 = 2\rho\, d\rho \cdot (2/\pi)\, d\theta$, since $R^2 = -\ell n\rho^2$ and $e^{-R^2} = \rho^2$. By C2, we may sample the latter two densities for ρ,θ and transform to ν_1,ν_2. But from the equivalence $2\rho\, d\rho \cdot (2/\pi)\, d\theta = (4/\pi) \cdot dx\, dy$ on the unit disk in quadrant I, we may sample the disk for $\hat{x},\hat{y}$ uniformly in area, and take the corresponding polar values $\rho = (\hat{x}^2 + \hat{y}^2)^{1/2}$, and θ with $\cos\theta = \hat{x}/\rho$, $\sin\theta = \hat{y}/\rho$. Since this is just what R1 does, and $\rho = S^{1/2}$, one has $R = (-2\ell n\rho)^{1/2} = (-\ell n\rho^2)^{1/2}$

$= (-\ell n S)^{1/2}$, and the rule follows from the transformation to (ν_1,ν_2). (Box, Muller, Marsaglia.) See also R10.

C52. $q(u) = u^{-1} \exp\{-(\ell n^2 u)/2b\}/(2\pi b)^{1/2}$; $(0,\infty)$, $b > 0$.

R_x. Sample $e^{-x^2}/\pi^{1/2}$ for x on $(-\infty,\infty)$ by C59 or R11. Set $u = e^{x(2b)^{1/2}}$.

J. For this substitution one finds that $q(u)\,du = e^{-x^2}\,dx/\pi^{1/2}$ on $(-\infty,\infty)$, and the rule follows from C2.

Note. The "log-normal" density q(u) is the density for the function e^x under the normal density $e^{-x^2/2b}/(2\pi b)^{1/2}$. For,

$$\frac{d}{du}\int_{\{e^x \le u\}} e^{-x^2/2b}\,dx/(2\pi b)^{1/2} = \frac{d}{du}\int_{-\infty}^{\log u} e^{-x^2/2b}\,dx/(2\pi b)^{1/2} = q(u) \text{ as}$$

above. (See C5.)

C53. $p(x) = (2\pi)^{-1/2}\lambda^{-1}\delta\left[1 + \left(\frac{x-\xi}{\lambda}\right)^2\right]^{-1/2} \exp\left\{-(1/2)\left[\gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right)\right]^2\right\}$; $(-\infty,\infty)$, ξ,γ arbitrary, λ, $\delta > 0$.

R_x. Sample $e^{-y^2}/\pi^{1/2}$ for y on $(-\infty,\infty)$ by C59 or R11.
Set $x = \xi + \lambda \sinh((2^{1/2}y - \gamma)/\delta)$.

J. The preceding function $x = x(y)$ increases from $x = -\infty$ to $x = \infty$ for y on $(-\infty,\infty)$. Moreover,

$$1 + \left(\frac{x-\xi}{\lambda}\right)^2 = 1 + \sinh^2((2^{1/2}y - \gamma)/\delta) = \cosh^2((2^{1/2}y - \gamma)/\delta), \text{ and}$$

$dx = \lambda \cosh((2^{1/2}y - \gamma)/\delta)(2^{1/2}/\delta)\,dy$. Hence $p(x)\,dx = e^{-y^2}\,dy/\pi^{1/2}$, and the rule follows from C2.

C54. $p(x) = (2\pi)^{-1/2}\lambda^{-1}\delta\left(\frac{x-\xi}{\lambda}\right)^{-1}\left(1 - \frac{x-\xi}{\lambda}\right)^{-1} \exp\left\{-(1/2)\left[\gamma + \delta\ell n\left(\frac{x-\xi}{\xi+\lambda-x}\right)\right]^2\right\}$; $(\xi,\xi+\lambda)$, ξ, γ arbitrary, λ, $\delta > 0$.

R_x. Sample $e^{-y^2}/\pi^{1/2}$ for y on $(-\infty,\infty)$ by C59 or R11. Set $x = \xi + [\lambda/(1 + E)]$, where $E = e^{-Y}$ and $Y = (2^{1/2}y - \gamma)/\delta$.

J. First note in p(x) that $(x - \xi)/((\xi + \lambda) - x) > 0$ for $\xi < x < \xi + \lambda$. The function $x = x(y) = \xi + [\lambda/(1 + e^{-Y})]$ increases from $x = \xi$ to $x = \xi + \lambda$

for y on $(-\infty,\infty)$. Moreover, $(x - \xi)/\lambda = 1/(1 + e^{-Y})$, $1 - ((x - \xi)/\lambda)$ $= e^{-Y}/(1 + e^{-Y})$, and $(x - \xi)/(\xi + \lambda - x) = \frac{(x - \xi)}{\lambda}\Big/\left(1 - \frac{x - \xi}{\lambda}\right) = e^{Y}$.

Finally, $dx = \lambda(1 + e^{-Y})^{-2}e^{-Y}(2^{1/2}/\delta)\,dy$.

Substitution shows that $p(x)\,dx = e^{-y^2}\,dy/\pi^{1/2}$, and the rule follows from C2.

C55. $q(u) = \exp(-\ell n^2 u)/e^{1/4}\pi^{1/2}$; $(0,\infty)$.

R_x. Sample $e^{-y^2}/\pi^{1/2}$ for y on $(-\infty,\infty)$ by C59 or R11.
Set $u = e^{y+(1/2)}$. (Cashwell.)

J. The function $u = e^{y+(1/2)}$ increases from $u = 0$ to $u = \infty$ for y on $(-\infty,\infty)$, and $q(u)\,du = e^{-y^2}\,dy/\pi^{1/2}$. The rule follows from C2.

C56. $p(x) = (x - \theta)^{-1}(2\pi b)^{-1/2}\exp\{-[\ell n(x - \theta) - \zeta]^2/2b\}$; (θ,∞), $b > 0$, θ, ζ arbitrary.

R_x. Sample $e^{-y^2}/\pi^{1/2}$ for y on $(-\infty,\infty)$ by C59 or R11. Set $x = \theta + \exp\{\zeta + (2b)^{1/2}y\}$.

J. The preceding function $x = x(y)$ increases from $x = \theta$ to $x = \infty$ for y on $(-\infty,\infty)$. For $Y \equiv \zeta + (2b)^{1/2}y$, one has $x - \theta = e^{Y}$, $\ell n(x - \theta) = Y$, and $dx = e^{-Y}(2b)^{1/2}\,dy$. Hence $p(x)\,dx = e^{-y^2}\,dy/\pi^{1/2}$ and the rule follows from C2.

Note. For $\theta = \zeta = 0$, C56 is the log-normal density of C52.

C57. $s(w) = (2\pi\sigma^2)^{-1/2}\cosh(\xi w/\sigma^2)e^{-(w^2+\xi^2)/2\sigma^2}$; $(-\infty,\infty)$, ξ arbitrary, $\sigma > 0$.

R_x. Sample $e^{-\nu^2}/\pi^{1/2}$ for ν on $(-\infty,\infty)$ by C59 or R11. Set $z = \xi + (2\sigma^2)^{1/2}\nu$, and $w = \pm z$ with probability 1/2.

J. One first notes that $s(w) = (1/2)((g/w) + g(-w))$, where $g(z) = e^{-(z-\xi)^2/2\sigma^2}/(2\pi\sigma^2)^{1/2}$ is a density on $(-\infty,\infty)$. Hence by C131, we may sample g(z) for z on $(-\infty,\infty)$, and set $w = \pm z$ as in the rule. But for $z = \xi + (2\sigma^2)^{1/2}\nu$, one has $g(z)\,dz = e^{-\nu^2}\,d\nu/\pi^{1/2}$, and the rule follows from C2.

C58. $h(\hat{w}) = (2/\pi\sigma^2)^{1/2} \cosh(\xi\hat{w}/\sigma^2)e^{-(\hat{w}^2+\xi^2)/2\sigma^2}$; $(0,\infty)$, ξ arbitrary, $\sigma > 0$.

R_x. Sample $e^{-\nu^2}/\pi^{1/2}$ for ν on $(-\infty,\infty)$ by C59 or R11. Set $\hat{w} = |\xi + (2\sigma^2)^{1/2}\nu|$.

J. The rule follows from C27 and C57. For, $h(\hat{w}) = 2s(\hat{w})$, where $s(w) = (2\pi\sigma^2)^{-1/2} \cosh(\xi w/\sigma^2)e^{-(w^2+\xi^2)/2\sigma^2}$ is the symmetric density of C57. By C27, we can sample $s(w)$ for w on $(-\infty,\infty)$ as in C57, and set $\hat{w} = |w|$. The rule follows.

C59. $p(x) = e^{-x^2}/\pi^{1/2}$; $(-\infty,\infty)$.

R_x. Sample $p(\nu_1) = 2e^{-\nu_1^2}\Big/\pi^{1/2}$ for ν_1 on $(0,\infty)$ by C51. Set $x = \pm\,\nu_1$ with probability 1/2.

J. The rule follows from C28. See also R11.

C60. $q(y) = e^{-y^2/2}\Big/(2\pi)^{1/2}$; $(-\infty,\infty)$.

R_x. Sample $p(x)$ for x on $(-\infty,\infty)$ as in C59. Set $y = 2^{1/2}x$.

J. Under this transformation, one has $q(y)\,dy = p(x)\,dx$ as in C59, and the rule follows from C2. See also R9.

C61. $q(u) = 2u^{2n-1}e^{-u^2}/\Gamma(n)$, $(0,\infty)$, $n \in \{1/2, 3/2, \ldots\}$, $2n$ odd. (For $n = 1/2$, use C51.)

R_x. Define $h = n - 1/2$, $h \in \{0,1,2,\ldots\}$, i.e., $2n = 2h + 1$. Sample $2e^{-\tau^2}\Big/\pi^{1/2}$ for τ on $(0,\infty)$ by C51 or R10. Set $u = \{-\ln(r_1 \ldots r_h) + \tau^2\}^{1/2}$.

J. For arbitrary $N = 1,2,3,\ldots$, one can write the density $q(u) = 2u^{N-1}e^{-u^2}\Big/\Gamma(N/2) = F(u)A(u)$, where $F(u) = 2^N e^{-u^2}\Big/\pi^{N/2}$, and $A(u) = \pi^{N/2}u^{N-1}/2^{N-1}\Gamma(N/2)$. By F8, $A(u) = dV/du$ for $V = \int_{\left\{\left(\sum_1^N \nu_i^2\right)^{1/2} \le u,\ \nu_i > 0\right\}} \prod_1^N d\nu_i$, and moreover, for the function $f(\nu) = \left(\sum_1^N \nu_i^2\right)^{1/2}$, one has $F(f(\nu))$ $= \prod_1^N \left(2e^{-\nu_i^2}/\pi^{1/2}\right)$, a product of N densities on $(0,\infty)$. Thus $q(u)$ is the density for the value u of the function $f(\nu) = \left(\sum_1^N \nu_i^2\right)^{1/2}$ under the density $\prod_1^N \left(2e^{-\nu_i^2}/\pi^{1/2}\right)$, and by C8, we may sample $2e^{-\nu^2}/\pi^{1/2}$ N times for the ν_i, as in C51, and set $u = \left(\sum_1^N \nu_i^2\right)^{1/2}$. Now, if N is even, as in C49, $N = 2h \geq 2$, this gives the h pairs (ν_1,ν_2), ..., (ν_{2h-1},ν_{2h}), with $\nu_{2i-1}^2 + \nu_{2i}^2 = -\ell n r_i$ in C51, $R_x 1$, and leads to the same sample $u = f(\nu)$ $= ((-\ell n r_1) + \ldots + (-\ell n r_h))^{1/2} = (-\ell n r_1 \ldots r_h)^{1/2}$ as that obtained in C49 on simpler grounds. However, for $N = 2n = 2h + 1 \geq 1$ odd, as here in C61, we get from C51,

(ν_1,ν_2), ..., (ν_{2h-1},ν_{2h}) and $\nu_{2h+1} = \tau$

and the rule above follows. R10 will also give τ.

<u>C62. $p(\nu) = 2\nu^{2n-1}/(e^{\nu^2} - 1)\zeta(n)\Gamma(n)$;</u> $(0,\infty)$, $n \in \{3/2, 5/2, \ldots\}$ (See F9E.)

R_x. Set $K = \min\left\{k;\ \sum_1^k (1/j^n) \geq r_0\zeta(n)\right\}$. Let $h = n - 1/2 \in \{1,2,\ldots\}$.

Sample $2e^{-\tau^2}/\pi^{1/2}$ for τ on $(0,\infty)$ by C51 or R10.

Set $\nu = \left\{K^{-1}\left(-\ell n \prod_1^h r_i + \tau^2\right)\right\}^{1/2}$.

J. Following C3, we write $p(\nu) = 2\zeta^{-1}(n)\Gamma^{-1}(n)\nu^{2n-1}e^{-\nu^2}\Big/\left(1 - e^{-\nu^2}\right)$

$$= 2\zeta^{-1}(n)\Gamma^{-1}(n) \sum_1^\infty \nu^{2n-1}e^{-j\nu^2} = \sum_1^\infty (1/j^n\zeta(n))\left(2j^n\nu^{2n-1}e^{-j\nu^2}\Big/\Gamma(n)\right).$$

Hence we set K as in the rule, and sample the density $2K^n\nu^{2n-1}e^{-K\nu^2}\Big/\Gamma(n)$ for ν on $(0,\infty)$. But for $\nu = u/K^{1/2}$, we have $2K^n\nu^{2n-1}e^{-K\nu^2}\ d\nu/\Gamma(n)$ $= 2u^{2n-1}e^{-u}\ du/\Gamma(n)$ as in C61. From this and C2 the rule follows.

C63. $q(u) = u^{n-1}/(e^u - 1)\zeta(n)\Gamma(n)$; $(0,\infty), n \in \{3/2,5/2, ...\}$. (See F9D.)

R_x. Set $K = \min\left\{k; \sum_1^k (1/j^n) \geq r_0\zeta(n)\right\}$. Let $h = n - 1/2$. Sample

$2e^{-\tau^2}\Big/\pi^{1/2}$ for τ on $(0,\infty)$ by C51 or R10. Set $u = K^{-1}\left(-\ell n \prod_1^h r_i + \tau^2\right)$.

J. For $u = \nu^2$, one has $q(u)\ du = p(\nu)\ d\nu$ as in C62. The rule follows from this and C2.

C64. $p(\nu) = \nu^{n-1}e^{-\nu}/\Gamma(n)$; $(0,\infty)$, $n \in \{1/2,3/2, ...\}$.

R_x. Define $h = n - 1/2$, $h \in \{0,1,2, ...\}$. Sample $2e^{-\tau^2}\Big/\pi^{1/2}$ for τ on $(0,\infty)$ by C51 or R10. Set $u = -\ell n \prod_1^h r_i + \tau^2$.

J. For $\nu = u^2$, one has $p(\nu)\ d\nu = 2u^{2n-1}e^{-u^2}\ du/\Gamma(n)$ as in C61. From this and C2, the rule follows. (For $n = 1/2$, $u = \tau^2$.)

C65. $g(y) = A^n y^{A-1} \ell n^{n-1}(1/y)/\Gamma(n)$; $(0,1)$, $A > 0$, $n \in \{1/2,1,3/2,2, ...\}$.

R_x. Sample $x^{n-1}e^{-x}/\Gamma(n)$ for x on $(0,\infty)$ by C45 or C64. Set $y = e^{-x/A}$.

J. For $y = e^{-x/A}$, one has $g(y)\ dy = x^{n-1}e^{-x}(-dx)/\Gamma(n)$, and the rule follows from C2.

C66. $g(t) = pt^{np-1}e^{-t^p}\Big/\Gamma(n)$; $(0,\infty)$, $p > 0$, $n \in \{1/2,1,3/2,2, ...\}$.

R_x. Sample $x^{n-1}e^{-x}/\Gamma(n)$ for x on $(0,\infty)$ by C45 or C64. Set $t = x^{1/p}$.

J. For $t = x^{1/p}$ one has $g(t)\ dt = x^{n-1}e^{-x}\ dx$, and the rule follows from C2. (For $p = 2$, see C49, C61.)

$\underline{\text{C67.}\quad p(t) = (\rho^{\phi}/\sigma\Gamma(\phi))e^{-\phi t/\sigma}\ \exp(-\rho e^{-t/\sigma})}$; $(-\infty,\infty)$, $\rho,\sigma,\phi > 0$.

R_x. Sample $w^{\phi-1}e^{-w}/\Gamma(\phi)$ for w on $(0,\infty)$ by C45, C64, or R27. Set $t = -\sigma \ln(w/\rho)$.

J. For the preceding (t,w) transformation, one has $p(t)\,dt = w^{\phi-1}e^{-w}(-dw)/\Gamma(\phi)$. The rule then follows from C2.

$\underline{\text{C68.}\quad p(x) = C^{-1}(1 + (x/a))^{ab}e^{-bx}}$; $(-a,\infty)$, $a,b > 0$, $C = ae^{ab}\Gamma(ab)/(ab)^{ab}$.

R_x. Define $n = ab + 1$. Sample $w^{n-1}e^{-w}/\Gamma(n)$ for w on $(0,\infty)$ by C45, C64, or R27. Set $x = (w - ab)/b$.

J. The function $x = (w - ab)/b$ increases from $x = -a$ to $x = \infty$ for w on $(0,\infty)$, and for this substitution, one sees that $p(x)\,dx = C^{-1}(w/ab)^{ab} \cdot e^{-w+ab} \cdot dw/b = w^{n-1}e^{-w}\,dw/\Gamma(n)$, and the rule follows from C2.

$\underline{\text{C69.}\quad p(x) = b^n/\Gamma(n)x^{n+1}e^{b/x}}$; $(0,\infty)$, $b,n > 0$.

R_x. Sample $w^{n-1}e^{-w}/\Gamma(n)$ for w on $(0,\infty)$ by C45, C64, or R27. Set $x = b/w$.

J. The function $x = b/w$ decreases from $x = \infty$ to $x = 0$ for w on $(0,\infty)$, and $p(x)\,dx = w^{n-1}e^{-w}(-dw)/\Gamma(n)$. The rule follows from C2.

$\underline{\text{C70.}\quad f(x) \equiv a^{2n}x^{-(n+1)}e^{-a^2/2x}/2^n\Gamma(n)}$; $(0,\infty)$, $a > 0$, $n \in \{1/2,1,3/2,2,\ \ldots\}$.

R_x. Sample $y^{n-1}e^{-y}/\Gamma(n)$ for y on $(0,\infty)$ by C45 or C64. Set $x = a^2/2y$.

J. For $x = a^2/2y$, one has $f(x)\,dx = y^{n-1}e^{-y}(-dy)/\Gamma(n)$ on $(0,\infty)$, and the rule follows from C2.

$\underline{\text{C71.}\quad s(x) = (2/\pi^2)\ x\ \text{csch}\ x}$; $(-\infty,\infty)$. (See F23.)

R_x. Set $K = \min\left\{k;\ \sum_1^k (2j - 1)^{-2} \geq r_0(\pi^2/8)\right\}$, $\xi = -(2K - 1)^{-1}\ \ln r_1r_2$, and $x = \pm\ \xi$ with probability 1/2.

J. Since $s(x)$ is symmetric, we may sample density $2s(\xi)$ for ξ on $(0,\infty)$, and let $x = \pm\ \xi$, as in the rule (C28). But $2s(\xi)$ may be written in the form

$$2s(\xi) = (8/\pi^2)\xi/(e^{\xi} - e^{-\xi}) = (8/\pi^2)\xi e^{-\xi}/(1 - e^{-2\xi}) = \sum_1^{\infty}\left\{(8/\pi^2) \cdot (2j - 1)^{-2} \cdot (2j - 1)^2\xi e^{-(2j-1)\xi}\right\}$$

as in C3. Hence we may set K as in the rule, and sample density $p(\xi) = (2K - 1)^2\xi e^{-(2K-1)\xi}$ for ξ on $(0,\infty)$. But for $\xi = (2K - 1)^{-1}\eta$, one has $p(\xi)\,d\xi = \eta e^{-\eta}\,d\eta$. The rule therefore follows from C45, with $n = 2$.

Note. $\zeta_u(2) = \sum_1^\infty 1/(2j-1)^2 = \pi^2/8$ by F9C.

C72. $q(u) = C^{-1}u^{n-1}E_N(u)$; $(0,\infty)$, $N \geq 0$, $n + N > 1$, $n \in \{1/2,1,3/2,2, \ldots\}$,

$$E_N(u) = \int_1^\infty \nu^{-N}e^{-u\nu}\, d\nu, \quad C = \Gamma(n)/(n + N - 1). \quad \text{(See F18,19.)}$$

R_x. Sample $p_1(\nu_1) = (n + N - 1)\nu_1^{n+N-2}$ for $\nu_1 = r_1^{1/(n+N-1)}$ on (0,1) by C16, and $p_2(\nu_2) = \nu_2^{n-1}e^{-\nu_2}/\Gamma(n)$ for ν_2 on $(0,\infty)$ by C45 or C64. Set $u = \nu_1\nu_2$.

J. The rule follows from C9, since

$$\int_0^1 p_1(\nu_1)\nu_1^{-1}p_2(u/\nu_1)\, d\nu_1 = C^{-1}u^{n-1}\int_0^1 \nu_1^{N-2}e^{-u/\nu_1}\, d\nu_1 = C^{-1}u^{n-1}E_N(u)$$

$= q(u)$, by F18. Thus $q(u)$ is the density for the product $\nu_1\nu_2$ under the density $p_1(\nu_1)p_2(\nu_2)$ on $(0,1) \times (0,\infty)$.

Note. For $N = 0$, $n \in \{3/2,2,5/2, \ldots\}$, $E_N(u) = e^{-u}/u$, and $q(u) = u^{n-2}e^{-u}/\Gamma(n-1)$, which may be sampled for u on $(0,\infty)$ by C45 or C64.

C73. $q(u) = C^{-1}u^{n-1}K_N(u)$; $(0,\infty)$, $n > N \geq 0$, $n, N \in \{0,1,2, \ldots\}$ or n,

$$N \in \{1/2,3/2,5/2, \ldots\}, \quad K_N(u) = \int_0^\infty \cosh N\theta\, e^{-u\cosh\theta}\, d\theta, \quad C = 2^{n-2}\Gamma((n$$

$- N)/2)\Gamma((n + N)/2)$. (See F17.)

R_x1. Define $H = (n - N)/2$, $J = (n + N)/2$, $H, J \in \{1/2,1,3/2,2, \ldots\}$. Sample $p_1(\nu_1) = \nu_1^{H-1}e^{-\nu_1}/\Gamma(H)$, $p_2(\nu_2) = \nu_2^{J-1}e^{-\nu_2}/\Gamma(J)$ for ν_1,ν_2 on $(0,\infty)$ by C45 or C64. Set $u = 2(\nu_1\nu_2)^{1/2}$.

J1. First note that, under the substitution $u = 2v^{1/2}$, one has $q(u)\, du = C^{-1}2^{n-1}v^{(n/2)-1}K_N(2v^{1/2})\, dv \equiv p(v)\, dv$, so it suffices to sample $p(v)$ for v on $(0,\infty)$, and set $u = 2v^{1/2}$ by C2. Second, using C9 and F13D, one sees that

$$\int_0^\infty p_1(\nu_1)\nu_1^{-1}p_2(v/\nu_1)\,d\nu_1 = \frac{v^{J-1}}{\Gamma(H)\Gamma(J)} \cdot \frac{2}{v^{N/2}} \cdot K_N(2v^{1/2})$$

$$= C^{-1}2^{n-1}v^{(n/2)-1}K_N(2v^{1/2}) = p(v).$$

Hence $p(v)$ is the density for $\nu_1\nu_2$ under $p_1(\nu_1)p_2(\nu_2)$, and we set $v = \nu_1\nu_2$ as in C9.

R_x2. Define $H = (n - N)/2$ and $K = N + (1/2)$. Sample $p(\xi) = \xi^{H-1}(1 - \xi)^{K-1}/B(H,K)$ for ξ on $(0,1)$ by C75, and $p_2(\nu_2) = \nu_2^{n+N-1}e^{-\nu_2}/\Gamma(n + N)$ for ν_2 on $(0,\infty)$ by C45. Set $u = \xi^{1/2}\nu_2$.

J2. For the density $p_1(\nu_1) = 2\nu_1^{n-N-1}(1 - \nu_1^2)^{N-(1/2)}/B(H,K)$ on $(0,1)$, and the density $p_2(\nu_2)$ above on $(0,\infty)$, one finds that $\int_0^1 p_1(\nu_1)\nu_1^{-1}p_2(u/\nu_1)\,d\nu_1$

$= (2^{N+1}\Gamma(N + (1/2))/(\Gamma(n + N)B(H,K)\Gamma(1/2))\; u^{n-1}K_N(u) = C^{-1}u^{n-1}K_N(u)$

$= q(u)$ as given. Here we have used the value of $K_N(u)$ in F13C, and the Legendre identity of F4D, with $m = (n + N)/2$ to identify the constant with $C^{-1} = 1/2^{n-2}\Gamma\left(\frac{n - N}{2}\right)\Gamma\left(\frac{n + N}{2}\right)$. It follows from C9 that $q(u)$ is the density for the product $\nu_1\nu_2$ under the density $p_1(\nu_1)p_2(\nu_2)$, so we may sample $p_1(\nu_1)$ for ν_1 on $(0,1)$, and $p_2(\nu_2)$ on $(0,\infty)$ and set $u = \nu_1\nu_2$. But for $\nu_1 = \xi^{1/2}$, we see that $p_1(\nu_1)\,d\nu_1 = \xi^{H-1}(1 - \xi)^{K-1}\,d\xi/B(H,K)$, so by C2 we may sample the latter for ξ on $(0,1)$ and set $\nu_1 = \xi^{1/2}$. The rule follows. (Noted by Kalos [24] for $n = N + 2$.)

<u>C74. $q(\nu) = C^{-1}\nu^{n-1}\Lambda e^{-\nu}/(1 - \Lambda^2e^{-2\nu})$;</u> $(0,\infty)$, $0 < \Lambda \le 1$, $n \in \{3/2,2,5/2,3, \ldots\}$, $C = \zeta_u(\Lambda,n)\Gamma(n)$, where $\zeta_u(\Lambda,n) = \sum_1^\infty \left\{\Lambda^{2j-1}/(2j - 1)^n\right\}$. (See F12.)

R_x. Set $K = \min\left\{k;\ \sum_1^k \Lambda^{2j-1}/(2j - 1)^n \ge r_0\zeta_u(\Lambda,n)\right\}$.

Sample $u^{n-1}e^{-u}/\Gamma(u)$ for u on $(0,\infty)$ by C45 or C64.
Set $\nu = u/(2K - 1)$.

J. We can write $q(\nu)$ in the form of C3, since $q(\nu) = C^{-1}\nu^{n-1}\sum_{1}^{\infty}\left\{\Lambda^{2j-1}\cdot e^{-(2j-1)\nu}\right\} = \sum_{1}^{\infty}(\Lambda^{2j-1}/(2j - 1)^n\zeta_u(\Lambda,n))((2j - 1)^n\nu^{n-1}e^{-(2j-1)\nu}/\Gamma(n))$.
Hence we may set K as in the rule, and sample the density $p_K(\nu) = (2K - 1)^n\nu^{n-1}e^{-(2K-1)\nu}/\Gamma(n)$ for ν on $(0,\infty)$. But for $\nu = u/(2K - 1)$, one has $p_K(\nu)\,d\nu = u^{n-1}e^{-u}\,du/\Gamma(n)$, and the rule follows from C2.

C75. $B(v) = v^{m-1}(1 - v)^{n-1}/B(m,n)$; (0,1).

$b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$; $(0,\infty)$.

$q(\theta) = 2\sin^{2m-1}\theta\cos^{2n-1}\theta/B(m,n)$; $(0,\pi/2)$.

$C(w) = pw^{mp-1}(1 - w^p)^{n-1}/B(m,n)$; (0,1).

$m,n \in \{1/2,1,3/2,2,\ldots\}$ in all.

R_x. Sample $x^{m-1}e^{-x}/\Gamma(m)$ and $y^{n-1}e^{-y}/\Gamma(n)$ for x,y on $(0,\infty)$ by C45 and/or C64. Set $v = x/(x + y)$, $z = v/(1 - v) = x/y$, $\theta = \arcsin(v^{1/2})$, $w = v^{1/p}$.

J. The densities $b(z)$, $q(\theta)$, $C(w)$ are equivalent to $B(v)$ under the indicated substitutions. In virtue of C2, it therefore suffices to sample $B(v)$ for v on (0,1). But under the transformation $x = uv$, $y = u(1 - v)$, with Jacobian $- u$, and inverse $u = x + y$, $v = x/(x + y)$, one sees that

$$\frac{x^{m-1}e^{-x}\,dx}{\Gamma(m)}\cdot\frac{y^{n-1}e^{-y}\,dy}{\Gamma(n)} = \frac{u^{m+n-1}e^{-u}\,du}{\Gamma(m + n)}\cdot\frac{v^{m-1}(1 - v)^{n-1}\,dv}{B(m,n)},$$

so by C2, we may sample the first two densities for x and y, and set $v = x/(x + y)$ as in the rule.

Note 1. The same rule results from taking $b(z)$ as the basic density, and noting that, for $p_1(\nu_1) = \nu_1^{n-1}e^{-\nu_1}/\Gamma(n)$, $p_2(\nu_2) = \nu_2^{m-1}e^{-\nu_2}/\Gamma(m)$, one has as in C9, $\int_0^\infty p_1(\nu_1)\nu_1p_2(z\nu_1)\,d\nu_1 = \frac{z^{m-1}}{\Gamma(m)\Gamma(n)}\int_0^\infty \nu_1^{m+n-1}e^{-(1+z)\nu_1}\,d\nu_1$

$$= \frac{z^{m-1}\Gamma(m+n)}{\Gamma(m)\Gamma(n)(1+z)^{m+n}} = z^{m-1}/(1+z)^{m+n}B(m,n) = q(z).$$ Thus $q(z)$ is the density for ν_2/ν_1 under the density $p_1(\nu_1)p_2(\nu_2)$. Note that $v_1 \simeq y$, $\nu_2 \simeq x$.

Note 2. For $n = 1$, see C15.

Note 3. For $m = 1/2$ or $n = 1/2$, $q(\theta)$ involves $\cos\theta$ or $\sin\theta$ only.

Note 4. For $m = 1/2 = n$, set $\theta = (\pi/2)r$, $v = \sin^2\theta$, $z = v/(1-v)$, or use R1 to obtain $v = \hat{r}_2^2/S$, $z = (\hat{r}_2/\hat{r}_1)^2$.

Note 5. The same rule results from the equivalence

$$\frac{2\xi^{2n-1}e^{-\xi^2}\,d\xi}{\Gamma(n)} \cdot \frac{2\eta^{2m-1}e^{-\eta^2}\,d\eta}{\Gamma(m)} = \frac{2\rho^{2(m+n)-1}e^{-\rho^2}}{\Gamma(m+n)} \cdot \frac{2\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta}{B(m,n)}$$

under the polar transformation $\xi = \rho\cos\theta$, $\eta = \rho\sin\theta$. We omit the details.

Note 6. The transformation $x = uv$, $y = u(1-v)$ is frequently employed in the sequel. See R27.

C76. $q(x) = (x-a)^{m-1}(b-x)^{n-1}/(b-a)^{m+n-1}B(m,n)$; (a,b), $a < b$, $m,n > 0$.

R_x. Sample $B(v) = v^{m-1}(1-v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C75 or R28. Set $x = a + (b-a)v$.

J. For this (x,v) substitution, one has

$$q(x)\,dx = \frac{(b-a)^{m-1}v^{m-1}(b-a)^{n-1}(1-v)^{n-1}(b-a)\,dv}{(b-a)^{m+n-1}B(m,n)}$$

$$= v^{m-1}(1-v)^{n-1}\,dv/B(m,n).$$

The rule then follows from C2. Note. See C145.

C77. $p(x) = (b-a)^{Q-R-1}(x-b)^R/(x-a)^Q B(Q-R-1, R+1)$; (b,∞), $b > a$, $Q > R+1 > 0$.

R_x. Define $m = R+1$, $n = Q-R-1$. Sample $b(z) = z^{m-1}/(1+z)^{m+n}B(m,n)$ for z on $(0,\infty)$ by C75 or R28. Set $x = b + (b-a)z$.

J. For this (x,z) substitution, one finds that $p(x)\,dx = z^{m-1}\,dz/(1+z)^{m+n}\cdot B(m,n)$, and the rule follows from C2.

C78. $p(x) = 1/B(m,m)(e^{x/2} + e^{-x/2})^{2m}$; $(-\infty,\infty)$, $m > 0$.

R_x. Sample $z^{m-1}/(1+z)^{2m}B(m,m)$ for z on $(0,\infty)$ by C75 or R28. Set $x = \ell nz$.

J. For $x = \ell nz$, one has $p(x)\,dx = e^{mx}\,dx/(e^x + 1)^{2m}B(m,m) = z^{m-1}\,dz/(1 + z)^{2m}B(m,m) = b(z)$ on $(0,\infty)$ as in C75, R28.

Note. $p(x) = \text{sech}^{2m}(x/2)/4^m B(m,m)$.

C79. $p(x) = \rho^m e^{-mx/\sigma}/\sigma B(m,n)(1 + \rho e^{-x/\sigma})^{m+n}$; $(-\infty,\infty)$, $\rho,\sigma,m,n > 0$.

R_x. Sample $b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$ for z on $(0,\infty)$ by C75 or R28. Set $x = -\sigma\ell n(z/\rho)$.

J. For this (x,z) substitution, we find that $p(x)\,dx = z^{m-1}(-dz)/(1 + z)^{m+n} \cdot B(m,n)$, and the rule follows from C2.

Note 1. See C148 for the case $\rho = \sigma = 1$, $m = N - k + 1$, $n = k$.

Note 2. For $\rho = \sigma = 1 = m = n$, $p(x)$ reduces to C120. For this, the rule in C79 would set $x = -\ell n(\ell nr_1/\ell nr_2)$, whereas that in C120 sets $x = \ell n(r_0/(1 - r_0))$. Moral: Never use a general method for a simpler special case.

C80. $e(x) = \rho^m e^{-mx/\sigma}(1 - \rho e^{-x/\sigma})^{n-1}/\sigma B(m,n)$; $(\sigma\ell n\rho,\infty)$, $\rho,\sigma,m,n, > 0$.

R_x. Sample $B(v) = v^{m-1}(1 - v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C75 or R28. Set $x = -\sigma\ell n(v/\rho)$.

J. The function $x = -\sigma\ell n(v/\rho)$ decreases from $x = \infty$ to $x = \sigma\ell n\rho$ for v on $(0,1)$, and $e(x)\,dx = v^{m-1}(1 - v)^{n-1}(-\,dv)/B(m,n)$. The rule follows from C2.

Note 1. See C151 for the case $\rho = \sigma = 1$, $m = N - k + 1$, $n = k$.

Note 2. For $\rho = \sigma = 1 = m = n$, $e(x) = e^{-x}$ on $(0,\infty)$.

C81. $p(x) = (1 - (x/a)^2)^{n-1}/aB(1/2,n)$; $(-a,a)$, $a,n > 0$.

R_x. Sample $v^{-1/2}(1 - v)^{n-1}/B(1/2,n)$ for v on $(0,1)$ by C75 or R28. Set $\hat{x} = av^{1/2}$, and $x = \pm\hat{x}$ with probability 1/2.

J. Since $p(x)$ is symmetric, we may sample $2p(\hat{x})$ for $\hat{x}$ on $(0,a)$ and set $x = \pm\hat{x}$ as above, by C28. But for $\hat{x} = av^{1/2}$, one has $2p(\hat{x})\,d\hat{x} = v^{-1/2}(1 - v)^{n-1}\,dv/B(1/2,n)$ for v on $(0,1)$, and the rule follows from from C2.

C82. $b(z) = z^{m-1}/(1 + z)B(m,1 - m)$; $(0,\infty)$, $0 < m < 1$.

R_x. Sample $b(z)$ as in C75 if $m = 1/2$, otherwise use R28.

J. Special case of $b(z) = z^{m-1}/(1 + z)^{m+n}/B(m,n)$ with $n = 1 - m$, and $B(m,n) = B(m,1 - m) = \Gamma(m)\Gamma(1 - m) = \pi/\sin m\pi$ (F4B).

C83. $q(x) = C^{-1}x(x - a)^{m-1}(b - x)^{n-1}$; (a,b), $0 < a < b$, $m, n > 0$, $C = (b - a)^{m+n-1}\left(\frac{mb + na}{m + n}\right) \cdot B(m,n)$.

R_x. Define $A_1 = a\Big/\left(\frac{mb + na}{m + n}\right)$. If $r_0 \leq A_1$, sample $v^{m-1}(1 - v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C75 or R28. If $r_0 > A_1$, sample $v^m(1 - v)^{n-1}/B(m + 1,n)$ for v on $(0,1)$ by C75 or R28. In either case, set $x = a + (b - a)v$.

J. For $x = a + (b - a)v$, we find that $q(x)\,dx = \{a_1(v) + a_2(v)\}\,dv$ where

$$a_1(v) = C^{-1}(b - a)^{m+n-1}av^{m-1}(1 - v)^{n-1},$$

$$A_1 = \int_0^1 a_1(v)\,dv = a\Big/\left(\frac{mb + na}{m + n}\right),$$

$$a_1(v)/A_1 = v^{m-1}(1 - v)^{n-1}/B(m,n),$$

$$a_2(v) = C^{-1}(b - a)^{m+n}v^m(1 - v)^{n-1},$$

$$A_2 = m(b - a)/(mb + na),$$

$$a_2(v)/A_2 = v^m(1 - v)^{n-1}/B(m + 1,n).$$

The rule therefore follows from C2 and C3.

C84. $p(x) = C^{-1}x^{m-1}(1 - x)^{n-1}/(x + a)^{m+n}$; $(0,1)$, $a,m,n > 0$, $C = B(m,n)/(1 + a)^m a^n$. (See F5C.)

R_x. Sample $z^{m-1}/(1 + z)^{m+n}B(m,n)$ for z on $(0,\infty)$ by C75 or R28. Set $x = az/[1 + a(1 + z)]$.

J. For $x = y/(y + 1)$, which increases from $x = 0$ to $x = 1$, y on $(0,\infty)$, we find that $p(x)\,dx = C^{-1}y^{m-1}\,dy/[a + (1 + a)y]^{m+n}$, while for $y = az/(1 + a)$, the latter becomes $z^{m-1}dz/(1 + z)^{m+n}B(m,n)$. The rule then follows from C2, since the iterate of the two transformations $x = y/(y + 1)$ and $y = az/(1 + a)$ is $x = az/[1 + a(1 + z)]$, as in the rule.

C85. $q(x) = C^{-1}(a + x)^{m-1}(a - x)^{n-1}$; $(-a,a)$, $a,m,n > 0$, $C = (2a)^{m+n-1} \cdot B(m,n)$.

R_x. Sample $v^{m-1}(1 - v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C75 or R28. Set $x = a(2v - 1)$.

J. $x = a(2v - 1)$ increases from $x = -a$ to $x = a$ for v on $(0,1)$, and $q(x)\,dx = v^{m-1}(1 - v)^{n-1}\,dv/B(m,n)$. The rule follows from C2.

C86. $p(x) = F(x) + x^{-2}F(x^{-1})$; (0,1), $F(z)$ density on $(0,\infty)$.

R_x. Sample $F(z)$ for z on $(0,\infty)$. If $z \leq 1$, set $x = z$. If $z > 1$, set $x = 1/z$.

J. Under the rule, the probability of x on $(x, x + dx)$ in (0,1) is $F(x)\,dx + F(z)(-dz)$, $0 < x < 1$, $1 < z < \infty$, where $1/z = x$. But this is $F(x)\,dx + F(x^{-1})(x^{-2}\,dx) = p(x)\,dx$, as required.

C87. $p(x) = (x^{m-1} + x^{n-1})/(1 + x)^{m+n}B(m,n)$; (0,1), $m,n > 0$. (See F5.E.)

R_x. Sample $b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$ for z on $(0,\infty)$ by C75 or R28. If $z \leq 1$, set $x = z$. If $z > 1$, set $x = 1/z$.

J. For the density $b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$ on $(1,\infty)$, one has $x^{-2}b(x^{-1}) = x^{n-1}/(1 + x)^{m+n}B(m,n)$, and the rule follows from C86.

C88. $p(x) = 1/2\, m\lambda B(m,n)\{1 + |(x - \theta)/\lambda|^{1/m}\}^{m+n}$; $(-\infty,\infty)$, θ arbitrary, $\lambda,m,n > 0$.

R_x. Sample $b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$ for z on $(0,\infty)$ by C75 or R28. Set $\hat{w} = z^m$ and $w = \pm\hat{w}$ with probability 1/2. Finally, set $x = \theta + \lambda w$.

J. For $x = \theta + \lambda w$, one finds that $p(x)\,dx = dw/2mB(m,n)\{1 + |w|^{1/m}\}^{m+n} \equiv s(w)\,dw$ on $(-\infty,\infty)$. Hence we may sample $s(w)$ for w and set $x = \theta + \lambda w$. But $s(w)$ is symmetric on $(-\infty,\infty)$ so by C28, we sample density $2s(\hat{w})$ for $\hat{w}$ on $(0,\infty)$, and set $w = \pm\hat{w}$, as in the rule. Since $2s(\hat{w})\,d\hat{w} = d\hat{w}/mB(m,n)\{1 + \hat{w}^{1/m}\}^{m+n}$, we see that, for $\hat{w} = z^m$, we have $2s(\hat{w})\,d\hat{w} = z^{m-1}\,dz/(1 + z)^{m+n}B(m,n)$. Hence we may sample the latter for z on $(0,\infty)$, and set $\hat{w} = z^m$. The rule follows.

C89. $p(\nu_1, \ldots, \nu_N) = \pi^{-N/2} e^{-\sum_1^N \nu_i^2}$; $-\infty < \nu_i < \infty$, $N \in \{1,2,3, \ldots\}$.

R_x. For even $N = 2h$, obtain the h pairs

$(\nu_1,\nu_2), \ldots, (\nu_{2h-1},\nu_{2h})$

by sampling $e^{-\nu^2}/\pi^{1/2}$ on $(-\infty,\infty)$ by C59 via C51, where (note) the samples are produced in pairs. For odd $N = 2h + 1$, obtain also the additional pair (ν_{2h+1},ν_{2h+2}). In either case, set $\nu = (\nu_1, \ldots, \nu_N)$. See also R11.

J. Since $p(\nu) = \prod_{1}^{N} \left(e^{-\nu_i^2} / \pi^{1/2} \right)$, the rule follows from C6.

<u>Note</u>. The density for the <u>value</u> of the function

$$\chi = f(\nu) = \left(\sum_{1}^{N} \nu_i^2 \right)^{1/2} \quad \text{is } 2u^{N-1}e^{-u^2} / \Gamma(N/2), \text{ as seen in C93 with } b = 1/2.$$

<u>C90. $p(\Omega) = \Gamma(N/2)/2\pi^{N/2}$.</u> (See F8).

R_x. Obtain vector $\nu = (\nu_1, \ldots, \nu_N)$ as in C89. Set $\Omega = (\omega_1, \ldots, \omega_N)$, where

$$\omega_i = \nu_i/u \text{ and } u = \left(\sum_{1}^{N} \nu_i^2 \right)^{1/2} .$$

J. The rule determines a uniformly distributed direction Ω in N-space, equivalently, a point on the unit sphere $|\Omega| = 1$. See also R3,5,6.

<u>Note</u>. Observe that in C51, C59, the source of the ν_i, these components of ν are produced in pairs (ν_{2i-1}, ν_{2i}), with $\nu_{2i-1}^2 + \nu_{2i}^2 = -\ln r_i$ or $-\ln S_i$. This saves time in computing u.

<u>C91. $p(\nu_1, \ldots, \nu_N) = F\left(\left(\sum_{1}^{N} \nu_i^2 \right)^{1/2} \right)$</u>; $-\infty < \nu_i < \infty$, $N \in \{1,2, \ldots\}$. (Radially symmetric density.)

R_x. Sample the density $q(u) = \frac{2\pi^{N/2}}{\Gamma(N/2)} u^{N-1} F(u)$ for the radius u on $(0,\infty)$. Sample the unit sphere $|\Omega| = 1$ for $\Omega = (\omega_1, \ldots, \omega_N)$ by C90 or R3,5,6. Set $\nu_i = u\omega_i$.

J. To see that the above q(u) is indeed the density for the value u of the radius $f(\nu) = \left(\sum_{1}^{N} \nu_i^2 \right)^{1/2}$ under the density $p(\nu_1, \ldots, \nu_N)$, we note that q(u) may be written in the form $q(u) = F(u)A(u)$, with the given F(u), and

$A(u) = \frac{2\pi^{N/2}}{\Gamma(N/2)} u^{N-1} = dV/du$, where $V(u) = \int_{\left\{\left(\sum_1^N \nu_i^2\right)^{1/2} \leq u\right\}} \prod_1^N d\nu_i$ is the volume of the full N-sphere of radius u, as in F8. Since for $f(\nu) = \left(\sum_1^N \nu_i^2\right)^{1/2}$, we have $F(f(\nu)) = p(\nu_1, \ldots, \nu_N)$, it follows from C8, Note 3, that q(u) is the density for the radius u, as stated.

Note. For $p(\nu_1, \ldots, \nu_N) = \pi^{-N/2} e^{-\sum_1^N \nu_i^2}$, use C89.

C92. $p(s) = s^{(N/2)-1} e^{-s/2b}/(2b)^{N/2}\Gamma(N/2)$; $(0,\infty)$, $b > 0$, $N \in \{1,2, \ldots\}$.

R_x. Sample $w^{(N/2)-1}e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C45 or C64. Set $s = 2bw$.

J. For $s = 2bw$, one has $p(s)\, ds = w^{(N/2)-1} e^{-w}\, dw/\Gamma(N/2)$, and the rule follows from C2.

Note. p(s) is the density for the value of the function $s = \sum_1^N \nu_i^2 (= \chi^2)$ under the density $\prod_1^N e^{-\nu_i^2/2b}\Big/(2\pi b)^{1/2}$; $-\infty < \nu_i < \infty$. (See C93.)

C93. $q(u) = 2u^{N-1}e^{-u^2/2b}\Big/(2b)^{N/2}\Gamma(N/2)$; $(0,\infty)$, $b > 0$, $N \in \{1,2, \ldots\}$.

R_x. Sample $w^{(N/2)-1}e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C45 or C64. Set $u = (2bw)^{1/2}$.

J. For $u = s^{1/2}$, one has $q(u)\, du = p(s)\, ds$ as in C92.

Note. q(u) is the density for the value of the function $u = \left(\sum_1^N \nu_i^2\right)^{1/2}$ $(= \chi)$ under the density $\prod_1^N e^{-\nu_i^2/2b}\Big/(2\pi b)^{1/2}$, $-\infty < \nu_i < \infty$.

For, we note that q(u) is the form F(u)A(u), where $F(u) = e^{-u^2/2b}\Big/(2\pi b)^{N/2}$, and $A(u) = 2\pi^{N/2}u^{N-1}/\Gamma(N/2) = dV/du$, for V(u)

$$= \int_{\left\{\left(\sum_1^N \nu_i^2\right)^{1/2} \le u\right\}} \prod_1^N d\nu_i,$$ the volume of the full N-sphere of radius u, as

in F8. Moreover, for the function $f(\nu) = \left(\sum_1^N \nu_i^2\right)^{1/2}$, we see that

$F(f(\nu)) = \prod_1^N e^{-\nu_i^2\ 2b} \Big/ (2\pi b)^{1/2}$. Thus, by C8, the given q(u) is the

density for $\left(\sum_1^N \nu_i^2\right)^{1/2}$ under the latter density.

C94. $\underline{q_0(u) = N^{N/2}\mu^{(N/2)-1}e^{-N\mu/2b}/(2b)^{N/2}\Gamma(N/2)}$; $(0,\infty)$, $b > 0$, $N \varepsilon \{1,2, \ldots\}$.

R_x. Sample $w^{(N/2)-1}e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C45 or C64. Set $\mu = 2bw/N$.

J. For $\mu = s/N$, one has $q_0(\mu)\ d\mu = p(s)$ as in C92.

Note. $q_0(\mu)$ is the density for the function $\mu = \left(\sum_1^N \nu_i^2\right) \Big/ N$ ($= \chi^2/N$, mean square) under the density $\prod_1^N e^{-\nu_i^2/2b} \Big/ (2\pi b)^{1/2}$, $-\infty < \nu_i < \infty$.

C95. $\underline{p_1(\rho) = 2N^{N/2}\rho^{N-1}e^{-N\rho^2/2b} \Big/ (2b)^{N/2}\Gamma(N/2)}$; $(0,\infty)$, $b > 0$, $N \varepsilon \{1,2, \ldots\}$.

R_x. Sample $w^{(N/2)-1}e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C45 or C64.

Set $\rho = (2bw/N)^{1/2}$.

J. For $\rho = (s/N)^{1/2}$, one has $p_1(\rho)\ d\rho = p(s)\ ds$ as in C92.

Note. $p_1(\rho)$ is the density for the function

$\rho = \left(\sum_1^N \nu_i^2/N\right)^{1/2} = (\chi^2/N)^{1/2}$ (root mean square) under the density

$$\prod_{1}^{N} e^{-\nu_i^2/2b} \Big/ (2\pi b)^{1/2}, \quad -\infty < \nu_i < \infty.$$

C96. $q_1(t) = \Gamma((N+1)/2)/(N\pi)^{1/2}\Gamma(N/2)(1 + (t^2/N))^{(N+1)/2}$; $(-\infty,\infty)$, $N \in \{1,2, \ldots\}$.

R_x1. Sample $w^{(N/2)-1}e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C45 or C64. Set $\rho = (w/N)^{1/2}$. Sample $e^{-\sigma^2}\Big/\pi^{1/2}$ for σ on $(-\infty,\infty)$ by C59 or R11. Set $t = \sigma/\rho$.

J1. For the density $p_1(\rho) = 2N^{N/2}\rho^{N-1}e^{-N\rho^2/2b}\Big/(2b)^{N/2}\Gamma(N/2)$ on $(0,\infty)$ of C95 and the density $p_2(\sigma) = e^{-\sigma^2/2b}\Big/(2\pi b)^{1/2}$ on $(-\infty,\infty)$, one finds, as in C9,

$$\int_0^\infty p_1(\rho)\rho p_2(t\rho)\, d\rho = q_1(t)$$

as given. (One makes the substitution $z = (N + t^2)\rho^2/2b$.) Thus $q_1(t)$ is the density for $t = \sigma/\rho$ under the density $p_1(\rho)p_2(\sigma)$, regardless of the value of b! The rule therefore follows from C9, C95, C59, where we have used the value $b = 1/2$.

Note. $q_1(t)$ is the density for the value of $t = \sigma\Big/\left(\sum_1^N \nu_i^2/N\right)^{1/2}$ (Student's t), where the ν_i and σ all have density $e^{-z^2/2b}\Big/(2\pi b)^{1/2}$ on $(-\infty,\infty)$, independently of b. For $N = 1$, use C98.

R_x2. Sample $x^{m-1}e^{-x}/\Gamma(m)$ with $m = 1/2$ by C64, and $y^{n-1}e^{-y}/\Gamma(n)$ with $n = N/2$ by C45 or C64 for x,y on $(0,\infty)$. Set $z = x/y$ and $\hat{t} = (Nz)^{1/2}$. Set $t = \pm\hat{t}$ with probability 1/2.

J2. Since $q_1(t)$ is symmetric on $(-\infty,\infty)$, we may sample the density $2q_1(\hat{t})$ for $\hat{t}$ on $(0,\infty)$, and set $t = \pm\hat{t}$ as in the rule (C28). But for $\hat{t} = N^{1/2}z^{1/2}$, one has $2q_1(\hat{t})\,d\hat{t} = \Gamma((N+1)/2)z^{-1/2}\,dz/\pi^{1/2}\Gamma(N/2)(1+z)^{(N+1)/2} = z^{-1/2}\,dz/(1+z)^{(N+1)/2}B(1/2,N/2) = z^{m-1}\,dz/(1+z)^{m+n}B(m,n)$ where $m = 1/2$, $n = N/2$. Thus we may sample the latter density for z on $(0,\infty)$ and set $\hat{t} = (Nz)^{1/2}$. But sampling for z by C75 gives $z = x/y$, where x and y are found as in the rule.

C97. $p(\zeta) = c^{2m-1} \Big/ \left(c^2 + (\zeta - \zeta_0)^2\right)^m B(1/2, m - (1/2))$; $(-\infty,\infty)$, $c > 0$, ζ_0 arbitrary, $m \in \{1,3/2,2,5/2, \ldots\}$.

R_x. Define $N = 2m-1 \in \{1,2,3, \ldots\}$. Sample $w^{(N/2)-1}e^{-w}/\Gamma(N/2)$ for w on $(0,\infty)$ by C45 or C64. Sample $e^{-\sigma^2} \Big/ \pi^{1/2}$ for σ on $(-\infty,\infty)$ by C59 or R11. Set $\zeta = \zeta_0 + (c\sigma/w^{1/2})$.

J. With N as defined and $\zeta = \zeta_0 + ct/N^{1/2}$, one finds that $p(\zeta)\,d\zeta = q_1(t)\,dt$, where $q_1(t)$ is the density for Student's t in C96. There, one sets $t = \sigma/\rho = \sigma/(w/N)^{1/2}$, so we now set $\zeta = \zeta_0 + (c/N^{1/2}) \cdot (\sigma N^{1/2}/w^{1/2}) = \zeta_0 + (c\sigma/w^{1/2})$, as in the rule.

C98. $q_1(t) = 1/\pi(1 + t^2)$; $(-\infty,\infty)$. (Cauchy density, case N = 1 of Student's t, C96.)

R_x1. Set $t = \tan(\pi/2)(2r_0 - 1)$.

J1. By C1, we set $r_0 = \int_{-\infty}^{t} q_1(t)\,dt = (1/\pi)(\arctan t + (\pi/2))$ and solve for t.

R_x2. Set $t = y/x$ as in R2.

J2. The ratio y/x in R2 is the tangent of an angle θ uniformly distributed on $(-\pi/2,\pi/2)$.

Note. One can show using $C9(q_2)$ that Cauchy's density is the density for the function $t = \nu_2/\nu_1$ under the density $p_1(\nu_1)p_2(\nu_2)$, where $p_i(\nu_i) = e^{-\nu_i^2} \Big/ (2\pi)^{1/2}$, $-\infty < \nu_i < \infty$, $i = 1,2$.

C99. $p(z) = 1/\pi\lambda[1 + ((z - \theta)/\lambda)^2]$; $(-\infty,\infty)$, $\lambda > 0$, θ arbitrary.

R_x1. Set $z = \theta + \lambda \tan(2r_0 - 1)(\pi/2)$.

J1. For $z = \theta + \lambda t$, one has $p(z)\,dz = dt/\pi(1 + t^2) = q_1(t)\,dt$, where $q_1(t)$ is Cauchy's density C98. The rule follows from this and C2.

R_x2. Set $z = \theta + \lambda(y/x)$, where y/x is obtained from R2.

J2. See C98, J2.

C100. $s(w) = [1 + ((w^2 + \theta^2)/\lambda^2)]/\pi\lambda[1 + 2((w^2 + \theta^2)/\lambda^2) + ((w^2 - \theta^2)/\lambda^2)^2]$; $(-\infty,\infty)$, $\lambda > 0$, θ arbitrary.

R_x. Sample p(z) for z on $(-\infty,\infty)$ as in C99. Set $w = \pm z$ with probability 1/2.

J. The rule follows from C131, since $s(w) = (1/2)(p(w) + p(-w))$.

<u>C101. $h(\hat{w}) = 2s(\hat{w})$</u>; $(0,\infty)$; s(w) the density of C100 on $(-\infty,\infty)$.

R_x. Sample s(w) for w on $(-\infty,\infty)$ as in C100. Set $\hat{w} = |w|$, i.e., $\hat{w} = |z|$ for z as obtained in C99.

J. The rule follows from C27.

<u>C102. $p(x) = (1/\pi)\ \text{sech}\ x$</u>; $(-\infty,\infty)$.

R_x. Set $y = -\ln \tan(\pi r_0/4)$, and $x = \pm y$ with probability 1/2.

J. Since p(x) is symmetric, we may sample density 2p(y) for y on $(0,\infty)$, and set $x = \pm y$ as in the rule (C28). But for $y = -\ln z$, one has $2p(y)\,dy = (4/\pi)e^{-y}\,dy/(1 + e^{-2y}) = (4/\pi)(-dz)/(1 + z^2)$ with z on (0,1). By C1, we set $r_0 = \int_0^z (4/\pi)\,dz/(1 + z^2) = (4/\pi)\arctan z$, and hence $z = \tan(\pi r_0/4)$. The rule follows.

<u>C103. $q(u) = 2/\pi(1 - u^2)^{1/2}$</u>; (0,1).

R_x1. Set $\theta = (\pi/2)r$ and $u = \sin\theta$.

J1. For $u = \sin\theta$, one has $q(u)\,du = (2/\pi)\,d\theta$.
Note also that for $u = v^{1/2}$, one obtains $q(u)\,du = v^{-1/2}(1 - v)^{-1/2}/B(1/2,1/2)$. Cf. C75, Note 4.

R_x2. Use R1 to obtain $S = \hat{r}_1^2 + \hat{r}_2^2 \le 1$, and set $u = \hat{r}_2/S^{1/2}$.

J2. u is the sine of an angle θ uniformly distributed on $(0,\pi/2)$. As an example of C5, note that

$$\frac{d}{du}\int_{\{\sin\theta \le u\}} (2/\pi)\,d\theta = (2/\pi)\frac{d}{du}\int_0^{\arcsin u} d\theta = 2/\pi(1 - u^2)^{1/2}.$$

<u>C104. $q(F) = (M/N)^{M/2}F^{(M/2)-1}/(1 + (MF/N))^{(M+N)/2}B(M/2,N/2)$</u>; $(0,\infty)$, $M,N \in \{1,2,\ldots\}$.

R_x. Define $m = M/2$, $n = N/2$. Sample $x^{m-1}e^{-x}/\Gamma(m)$ and $y^{n-1}e^{-y}/\Gamma(n)$ for x,y on $(0,\infty)$ by C45 and/or C64. Set $z = x/y$ and $F = Nz/M$.

J. For $F = Nz/M$, one finds $q(F)\,dF = z^{m-1}\,dz/(1 + z)^{m+n}B(m,n)$ on $(0,\infty)$ as in C75, which sets $z = x/y$, as in the rule.

Note. It follows from C9 that $q(F)$ is the density for the value of $F = y/x$ under the density $q_N(x)q_M(y)$, where $q_N(x) = N^{N/2}x^{(N/2)-1}e^{-Nx/2b}/(2b)^{N/2}\Gamma(N/2)$, and $q_M(y) = M^{M/2}y^{(M/2)-1}e^{-My/2b}/(2b)^{M/2}\Gamma(M/2)$ as in C94. Thus $q(F)$ is the density of the function

$$F = \left(\sum_1^M \mu_i^2/M\right)\Bigg/\left(\sum_1^N \nu_j^2/N\right).$$ (Snedecor's F) where all μ_i, ν_j have density $e^{-z^2/2b}/(2\pi b)^{1/2}$ on $(-\infty,\infty)$ (regardless of the value of b.)

C105. $p(E) = 2(M/N)^{M/2}E^{M-1}/(1 + (ME^2/N))^{(M+N)/2}B(M/2,N/2)$; $(0,\infty)$, $M,N \in \{1,2,3, \ldots\}$.

R_x. Define $m = M/2$, $n = N/2$. Sample $x^{m-1}e^{-x}/\Gamma(m)$ and $y^{n-1}e^{-y}/\Gamma(n)$ for x,y on $(0,\infty)$ by C45 and/or C64. Set $z = x/y$, $F = Nz/M$, and $E = F^{1/2}$.

J. For $E = F^{1/2}$, one has $p(E)\,dE = q(F)\,dF$, the density for Snedecor's F in C104. The rule follows from C104 and C2.

Note. $p(E)$ is the density for the value of the function

$$E = \left(\sum_1^M \mu_i^2/M\right)^{1/2}\Bigg/\left(\sum_1^N \nu_j^2/N\right)^{1/2}$$ (quotient of root mean squares), where all μ_i, ν_j have density $e^{-\xi^2/2b}\,(2\pi b)^{1/2}$ on $(-\infty,\infty)$, regardless of b.

C106. $q(y) = D_\xi^{-1}y^{n-1}e^{-\xi y}$; $(1,\infty)$, $\xi > 0$, $n \in \{1,2, \ldots\}$, $D_\xi = \xi^{-n}(n-1)!e^{-\xi}S_\xi$,

$$S_\xi = \sum_0^{n-1} \xi^\nu/\nu!$$ (See F3C.)

R_x. Set $K = \min\left\{k;\ \sum_0^k \xi^\nu/\nu! \geq r_0 S_\xi\right\}$ $(0 \leq K \leq n-1)$ and $y = 1 - \xi^{-1}\ln \prod_1^{n-K} r_i$.

J. Under the substitution $y = 1 + (u/\xi)$, one has $q(y)\,dy$

$$= \frac{1}{D_\xi \xi^n e^\xi}(u+\xi)^{n-1}e^{-u}\,du = \frac{1}{(n-1)!S_\xi}(u+\xi)^{n-1}e^{-u}\,du \equiv p(u)\,du,$$ so by

C2, we may sample $p(u)$ for u on $(0,\infty)$ and set $y = 1 + (u/\xi)$. But we may write $p(u) = \frac{1}{(n-1)!S_\xi}\sum_{0}^{n-1}\frac{(n-1)!}{\nu!(n-1-\nu)!}u^{n-1-\nu}\xi^{\nu}e^{-u}$

$= \sum_{0}^{n-1}(\xi^{\nu}/\nu!S_\xi)(u^{n-\nu-1}e^{-u}/(n-\nu-1)!)$ in the form of C3. We may therefore set K as in the rule and sample the density $u^{(n-K)-1}e^{-u}/(n-K-1)!$ for u on $(0,\infty)$ by C45, i.e., set $u = -\ln\prod_{1}^{n-K} r_i$, and the rule follows. Note that $n - K \geq 1$.

C107. $\bar{q}(z) = (D_\xi\eta^n)^{-1}z^{n-1}e^{-\xi z/\eta}$; (η,∞), $\xi,\eta > 0$, $n \in \{1,2,\ldots\}$, D_ξ defined as in C106. (See F3B.)

R_x. Sample $q(y)$ for y on $(1,\infty)$ by C106. Set $z = \eta y$.

J. For $z = \eta y$, one has $\bar{q}(z)\,dz = D_\xi^{-1}y^{n-1}e^{-\xi y}\,dy$ as in C106, and the rule follows from C2.

C108. $\bar{q}(z) = \hat{D}_\eta^{-1}ze^{-z}$; (η,∞), $\eta > 0$, $\hat{D}_\eta = e^{-\eta}(1+\eta)$.

R_x1. If $r_0(1+\eta) > 1$, set $z = \eta - \ln r_1$. Otherwise set $z = \eta - \ln r_1r_2$.

J1. This is the rule of C107 for the special case $n = 2$, $\xi = \eta$.

Note. The preceding method seems simpler and more direct than that of Carey and Drijard [3], given below, and of course allows extension to the more general case of C107.

R_x2. (Carey-Drijard). One follows the steps:

1. Set $P = e^{-\eta}$, $A = e^{-\eta}(1+\eta)$, $B = 1/(1+\eta)$.
2. Generate random numbers ρ_1,ρ_2.
3. If $\rho_1 \leq B$, go to (4). If $\rho_1 > B$, go to (5).
4. Set $r_1 = A\rho_1$, $r_2 = \rho_2$; go to (6).
5. Set $r_1 = P\exp\{(1+\eta)\rho_1 - 1\}$, $r_2 = \rho_2P/r_1$; go to (6).
6. Set $z = -\ln r_1r_2$.

J2. The justification of this rule is based on the following remarks.

a. To sample the density $ze^{-z}/\Gamma(2)$ on its full range $(0,\infty)$, one generates random numbers r_1,r_2, and sets $z = -\ln r_1r_2$ as in C45, where

(r_1,r_2) may be thought of as a point uniformly distributed in the unit square.

b. But for the residual density on (η,∞), one requires only such points (r_1,r_2) for which $z = -\ln r_1 r_2 > \eta$, i.e., $r_1 r_2 < e^{-\eta} \equiv P$. (One could of course accept only those points (r_1,r_2) lying below the hyperbola $r_1 r_2 = P$, but the efficiency would be poor for large η.)

c. The above (non-rejection) device is valid, since the two transformations in steps (4), (5) both have Jacobian $A = e^{-\eta}(1 + \eta)$, independent of ρ_1,ρ_2, and so transform the two rectangular areas of the full (ρ_1,ρ_2) unit square determined by the line $\rho_1 = B$ in a uniform way into the two required areas of the (r_1,r_2) unit square; the first a rectangle of base $e^{-\eta}$ and height 1, of area $e^{-\eta}$, and the second lying directly below the hyperbola $r_1 r_2 = e^{-\eta}$, with base $1 - e^{-\eta}$ and area $\eta e^{-\eta}$.

C109. $\overline{q}(\nu) = D^{-1}\nu^{n-1}\Lambda e^{-a\nu}/(1 - \Lambda^2 e^{-2a\nu})$; $(1,\infty)$, $a > 0$, $0 < \Lambda \leq 1$,

$n \in \{2,3, \ldots\}$, $D = \sum_1^\infty \Lambda^{2j-1} D_{(2j-1)a}$, where $D_\xi \equiv \xi^{-n}(n - 1)!\, e^{-\xi} S_\xi$, $S_\xi = \sum_0^{n-1} \xi^\nu/\nu!$ (See F16.)

R_x. Compute partial sums s_k of $s = \sum_1^\infty \Lambda^{2j-1}(2j - 1)^{-n} e^{-2ja} S_{(2j-1)a}$, where S_ξ is defined above. Set $K = \min\{k;\ s_k \geq r_0 s\}$. Use C106, with $\xi = (2K - 1)a$, to obtain y on $(1,\infty)$. Set $\nu = y$.

J. One can write $\overline{q}(\nu) = D^{-1}\nu^{n-1} \sum_1^\infty \Lambda^{2j-1} e^{-(2j-1)a\nu} = \sum_1^\infty \left(D^{-1}\Lambda^{2j-1} D_{(2j-1)a}\right) \cdot \nu^{n-1} e^{-(2j-1)a\nu}/D_{(2j-1)a}$. Since this sum is of the form in C3, we can set $K = \min\left\{k;\ \sum_1^k \Lambda^{2j-1} D_{(2j-1)a} \geq r_0 D\right\}$, and sample the density $\nu^{n-1} e^{-(2K-1)a\nu}/D_{(2K-1)a}$ for ν on $(1,\infty)$ by C106, with $\xi = (2K - 1)a$. Note that the inequality involved in setting K is

$$\sum_{1}^{k} \Lambda^{2j-1}(2j - 1)^{-n} a^{-n}(n - 1)!\, e^{-(2j-1)a} S_{(2j-1)a} \geq r_0 \left\{ \sum_{1}^{\infty} \left[\Lambda^{2j-1}(2j-1)^{-n} \right. \right.$$

$$\left. \left. \cdot\, a^{-n}(n - 1)!\; e^{-(2j-1)a} S_{(2j-1)a} \right] \right\}$$, and the common factor $a^{-n}(n - 1)!\; e^{a}$, independent of j, has been deleted in the rule.

C110. $s(u) = \begin{cases} u; (0,1) \\ 2 - u;\ (1,2). \end{cases}$

R_x. Set $u = r_1 + r_2$.

J. For the uniform densities $p_1(\nu_1) \equiv 1 \equiv p_2(\nu_2)$, ν_i on (0,1), it is obvious geometrically that

$$P\{\nu_1 + \nu_2 \leq u\} = \begin{cases} u^2/2;\ (0,1) \\ 1 - (1/2)(2 - u)^2;\ (1,2). \end{cases}$$

Hence $\frac{d}{du} P\{\nu_1 + \nu_2 \leq u\} = s(u)$ as given, and the rule follows from C7.

<u>Note</u>. s(u) is the density for the sum of two random numbers.

(See Fig. C110.).

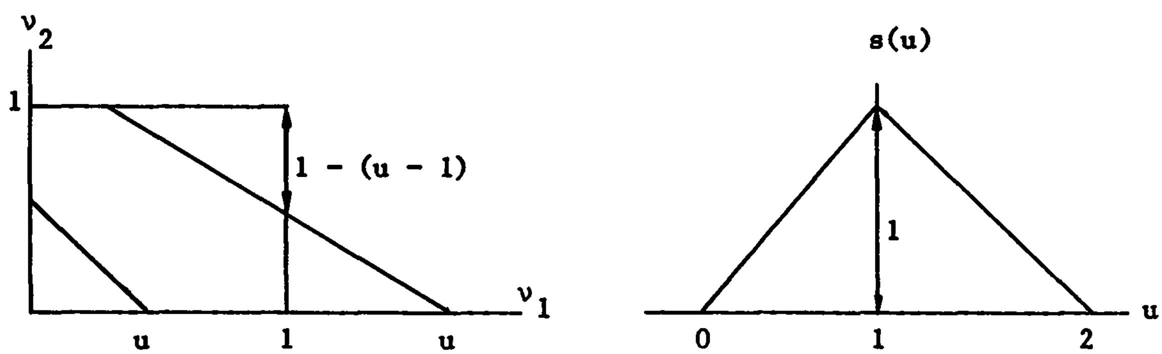


Fig. C110

C111. $t(x) = \begin{cases} 4(x - a)/(c - a)^2;\ (a,b) \\ 4(c - x)/(c - a)^2;\ (b,c), \end{cases}$ $a < c$, $b \equiv (a + c)/2$.

R_x. Set $x = a + (1/2)(c - a)(r_1 + r_2)$.

J. Under the transformation $x = a + (1/2)(c - a)u$, one has $t(x)\, dx = s(u)\, du$ as in C110, and the rule follows from C2.

<u>C112. $t(x) = 1 - |x|$; (-1,1).</u>

R_x. Set $x = r_1 - r_3$.

J. Special case of C111, with $a = -1$, $b = 0$, $c = 1$, where one sets $x = -1 + (1/2)(2)(r_1 + r_2) = -1 + r_1 + r_2 = r_1 - (1 - r_2) = r_1 - r_3$.

Note. $1 - |x|$ is the density for the difference of two random numbers.

C113. $q(x) = \begin{cases} a_1(x); & (a,b) \\ a_2(x); & (b,c),\ a < b < c. \end{cases}$

R_x. Define $A_1 = \int_a^b a_1(x)\,dx$, $A_2 = \int_b^c a_2(x)\,dx$, where $A_1 + A_2 = 1$. If $r_0 \leq A_1$, sample density $a_1(x)/A_1$ for x on (a,b). If $r_0 > A_1$, sample $a_2(x)/A_2$ for x on (b,c).

J. This may be regarded as a special case of C3, where $a_1(x)$, $a_2(x)$ are taken to be zero outside of (a,b) and of (b,c) respectively.

C114. $t(x) = \begin{cases} h(x - a)/(b - a); & (a,b) \\ h(c - x)/(c - b); & (b,c),\ a < b < c,\ h \equiv 2/(c - a). \end{cases}$

R_x. Define $A_1 = (b - a)/(c - a)$. For $r_0 \leq A_1$, set $x = a + (b - a)\max\{r_1,r_2\}$. If $r_0 > A_1$, set $x = c - (c - b)\max\{r_1,r_2\}$.

J. Following C113, we compute $A_1 = \int_a^b a_1(x)\,dx = (b - a)/(c - a)$ and $A_2 = \int_b^c a_2(x)\,dx = (c - b)/(c - a)$. Hence $a_1(x)/A_1 = 2(x - a)/(b - a)^2$ and $a_2(x)/A_2 = 2(c - x)/(c - b)^2$. For $x = a + (b - a)u$, and $x = c - (c - b)u$, respectively, one has $a_1(x)\,dx/A_1 = 2u\,du$, and $a_2(x)\,dx/A_2 = 2u(-du)$ on (0,1). By C15 (with $b = 1$, $m = 2$), we see that the density 2u on (0,1) may be sampled by setting $u = \max\{r_1,r_2\}$. Hence the rule follows from C113, C2, and C15. See also R18.

Note 1. Using C16, we could replace $\max\{r_1,r_2\}$ above by $r_3^{1/2}$.

Note 2. C114 is the most general "triangular" density. For $b = (a + c)/2$, the midpoint of (a,c), it reduces to C110, 111, or 112.

C115. $q(x) = \begin{cases} Aae^{ax}; & (-\infty,0) \\ Bbe^{-bx}; & (0,\infty),\ A,a,B,b > 0,\ A + B = 1. \end{cases}$

R_x. If $r_0 \le A$, set $x = a^{-1} \ln r_1$. If $r_0 > A$, set $x = -b^{-1} \ln r_1$.

J. Following C113, we find

$$A_1 \equiv \int_{-\infty}^{0} Aae^{ax}\, dx = A, \qquad A_2 \equiv \int_{0}^{\infty} Bbe^{-bx}\, dx = B,$$

$$a_1(x)/A_1 = ae^{ax};\ (-\infty,0) \qquad a_2(x)/A_2 = be^{-bx};\ (0,\infty).$$

For $x = -y$, $a_1(x)\, dx/A_1 = ae^{-ay}(-dy)$ on $(0,\infty)$. The rule therefore follows from C113, C29, and C2.

Note. $q(x)$ is continuous at $x = 0$ iff $Aa = Bb$.

C116. $q(x) = a_i(x);\ (x_i,x_{i+1}),\ i = 0,1,2,\ \ldots$ (Composite density.)

R_x. Define $A_i = \int_{x_i}^{x_{i+1}} a_i(x)\, dx$, where $\sum_0^\infty A_i = 1$. Set $K = \min\left\{k;\ \sum_0^k A_i \ge r_0\right\}$. Sample density $a_K(x)/A_K$ for x on (x_K,x_{K+1}).

J. Obvious application of C3, and generalization of C113.

$$\text{C117.}\quad q(x) = \begin{cases} a_0(x) = px/a^2;\ (0,a) \\ a_i(x) = pq^{i-1}[(1 + ip)a - px]/a^2;\ (ia,(i + 1)a),\ i = 1,2,\ \ldots \end{cases}$$

$a,p,q > 0$, $p + q = 1$.

R_x. Define $A_0 = p/2$. If $r_0 \le A_0$, set $x = ar_1^{1/2}$. If $r_0 > A_0$, set $K = \min\{k;\ (p/2)[1 + (1 + q)(1 + q + \ldots + q^{k-1})] \ge r_0;\ k \ge 1\}$. (See Note 2.) Set $x = (a/p)\{1 + Kp - [1 - (1 + q)pr_1]^{1/2}\}$.

J. Following C116, we compute $A_0 = \int_0^a a_0(x)\, dx = p/2$, $a_0(x)/A_0 = 2x/a^2$, and

for $i \ge 1$, $A_i = \int_{ia}^{(i+1)a} a_i(x)\, dx = pq^{i-1}(1 + q)/2$, $a_i(x)/A_i$

$= 2[(1 + ip)a - px]/a^2(1 + q)$. If $r_0 \leq A_0$, we set $r_1 = \int_0^x a_0(x)\, dx/A_0$ and find $x = ar_1^{1/2}$. If $r_0 > A_0$, we obtain $K \geq 1$ as in C116, and set $r_1 = \int_{Ka}^x a_K(x)\, dx/A_K$. Solving the latter by quadratic formula gives the x of the above rule.

Note 1. The choice of sign in solving the quadratic referred to for $x = (a/p)\{(1 + Kp) - [1 - (1 + q)pr_1]^{1/2}\}$ is in accordance with the fact that $r_1 = 0$ gives $x = Ka$, while $r_1 = 1$ gives $x = (K + 1)a$, as required.

Note 2. For $r_0 > A_0$, the rule demands that K be the least $k \geq 1$ for which $(p/2)[1 + (1 + q)(1 + q + \ldots + q^{k-1})] \geq r_0$. (The rule given in the second Sampler is wrong!) This may be simplified to the condition $q^K \leq 2(1 - r_0)/(1 + q) < q^{K-1}$.

Note 3. q(x) is a continuous broken line, with corners at x = 0, a, 2a, ..., and values $a_0(0) = 0$, $a_0(a) = a_1(a) = p/a$, and for $i \geq 2$, $a_{i-1}(ia) = a_i(ia) = pq^{i-1}/a$. q(x) has maximum at x = a, and thereafter decreases by a factor q at each step.

Note 4. Since the density $a_K(x)/A_K$ is linear, the method of C12 may be used instead of C1, which involves a square root.

C118. $p(x) = C^{-1}/(e^x + b + e^{-x})$; $(-\infty,\infty)$, $-2 < b < 2$, $C = B^{-1}(\pi/2 - \arctan(b/2B))$, where $B = (1 - (b^2/4))^{1/2}$.

R_x. Set $x = \ln\{-(b/2) + B((b/2B) + \tan(CBr_0))/(1 - (b/2B)\tan(CBr_0))\}$.

J. For $x = \ln y$, one sees that $p(x)\, dx = C^{-1}\, dy/(y^2 + by + 1) = C^{-1}\, dy/[(y + (b/2))^2 + B^2] \equiv q(y)\, dy$ on $(0,\infty)$. By C1, we set $r_0 = \int_0^y q(y)\, dy = C^{-1}B^{-1}\left\{\arctan \frac{y + (b/2)}{B} - \arctan \frac{b}{2B}\right\}$, and solve for $y = -(b/2) + B\tan\{\arctan(b/2B) + CBr_0\}$. Since $\tan(X + Y) = (\tan X + \tan Y)/(1 - \tan X \tan Y)$, the rule follows.

C119. $p(x) = C^{-1}/(e^x + e^{-x})$; $(-\infty,\infty)$, $C = \pi/2$.

R_x. Set $x = \ln \tan((\pi/2)r_0)$, or generate $\hat{r}_1, \hat{r}_2$ until $S \equiv \hat{r}_1^2 + \hat{r}_2^2 \le 1$, as in R1, and set $x = \ln(\hat{r}_2/\hat{r}_1) = \ln \hat{r}_2 - \ln \hat{r}_1$.

J. Special case b = 0 of C118.

Note. An equivalent form of C119 is $p(x) = (1/\pi)\,\mathrm{sech}\, x$.

C120. $p(x) = C^{-1}/(e^x + 2 + e^{-x})$; $(-\infty,\infty)$; $C = 1$.

R_x. Set $x = \ln r_0 - \ln(1 - r_0)$.

J. For $x = \ln y$, we find $p(x)\,dx = dy/(y^2 + 2y + 1) = dy/(y + 1)^2 \equiv q(y)\,dy$ on $(0,\infty)$. By C1, we set $r_0 = \int_0^y q(y)\,dy = y/(1 + y)$, giving $y = r_0/(1 - r_0)$. The rule follows.

Note. Other equivalent forms are $p(x) = 1/(e^{x/2} + e^{-x/2})^2 = (1/4)\mathrm{sech}^2(x/2) = e^x/(e^x + 1)^2 = e^{-x}/(1 + e^{-x})^2$.

C121. $p(x) = C^{-1}/(e^x + b + e^{-x})$; $(-\infty,\infty)$, $b > 2$, $C = D^{-1}\ln((b/2) + D)$, where $D = ((b^2/4) - 1)^{1/2}$.

R_x. Define $s = (b/2) + D$, $d = (b/2) - D$. Set $x = \ln(E - 1) - \ln(s - dE)$, where $E \equiv e^{2r_0 \ln s}$.

J. For $x = \ln y$, one has $p(x)\,dx = C^{-1}\,dy/(y^2 + by + 1) = C^{-1}\,dy/((y + (b/2))^2 - D^2) \equiv q(y)\,dy$ on $(0,\infty)$. By C1, we set $r_0 = \int_0^y q(y)\,dy = (1/2CD)\ln \frac{s(y + d)}{d(y + s)}$, and obtain $y = (E - 1)/(s - dE)$, which yields the rule. Note here that $sd = (b^2/4) - D^2 = 1$, and $CD = \ln((b/2) + D) = \ln s$.

C122. $q(y) = C^{-1}\alpha/(b + 2\cosh \alpha(y - y_0))$; $(-\infty,\infty)$ $\alpha > 0$, y_0 arbitrary, $b > -2$.

R_x. Sample p(x) for x on $(-\infty,\infty)$ by C118, 119, 120, or 121 (depending on the value of b). Set $y = y_0 + (x/\alpha)$.

J. For $y = y_0 + (x/\alpha)$, one has $q(y)\,dy = C^{-1}\,dx/(e^x + b + e^{-x}) = p(x)\,dx$ as in the cited references. The rule follows from C2.

C123. $r(t) = C^{-1}\alpha/t\{(t/t_0)^\alpha + b + (t/t_0)^{-\alpha}\}$, $(0,\infty)$, $\alpha, t_0 > 0$, $b > -2$.

R_x. Sample p(x) for x on $(-\infty,\infty)$ by C118, 119, 120, or 121 (depending on the value of b). Set $t = t_0 e^{x/\alpha}$.

J. For $t = t_0 e^{x/\alpha}$, one has $r(t)\,dt = C^{-1}\,dx/(e^x + b + e^{-x}) = p(x)\,dx$ as in the cited references. The rule follows from C2.

C124. $q(y) = \int_a^b dx\, f(x,y)$; (c,d), $f(x,y)$ density for $a < x < b$, $c < y < d$.

R_x. Sample the marginal density $p(x) = \int_c^d f(x,y)\,dy$ for x on (a,b).

For this x, sample the x-dependent y-density $p(y|x) = f(x,y)/p(x)$ for y on (c,d).

J. This is the continuous-continuous case of D24, where explanations are given.

C125. $q(y) = C^{-1}\left(e^{-ay^2} - e^{-by^2}\right)\Big/y^2$; $(0,\infty)$, $0 < a < b$, $C = \pi^{1/2}(b^{1/2} - a^{1/2})$.

R_x. Sample $\eta^{-1/2}e^{-\eta}/\pi^{1/2}$ for η on $(0,\infty)$ by C64. Set $y = \eta^{1/2}/[a^{1/2} + (b^{1/2} - a^{1/2})r_0]$.

J. For the density $f(x,y) = C^{-1}e^{-xy^2}$ on $(a,b) \times (0,\infty)$, one has the y-marginal density $\int_a^b dx\, f(x,y) = C^{-1}\int_a^b dx\, e^{-y^2 x} = q(y)$ as given above.

Following C124, we compute the x-marginal density $p(x) = \int_0^\infty f(x,y)\,dy$ $= C^{-1}\int_0^\infty e^{-xy^2}\,dy = x^{-1/2}/2(b^{1/2} - a^{1/2})$, where we have made the substitution $y = \eta^{1/2}/x^{1/2}$. Hence the x-dependent y-density $p(y|x)$ $= f(x,y)/p(x) = (2/\pi^{1/2})x^{1/2}e^{-xy^2}$. By C124, we may sample p(x) for x on (a,b), and, for this x, sample $p(y|x)$ for y on $(0,\infty)$. For the first, we set $r_0 = \int_a^x p(x)\,dx$ by C1, obtaining $x = [a^{1/2} + (b^{1/2} - a^{1/2})r_0]^2$.

Moreover, for $y = \eta^{1/2}/x^{1/2}$, one has $p(y|x) = \eta^{-1/2}e^{-\eta}\,d\eta/\pi^{1/2}$. Hence we sample $\eta^{-1/2}e^{-\eta}/\pi^{1/2}$ for η on $(0,\infty)$ by C64, and set $y = \eta^{1/2}/x^{1/2}$ $= \eta^{1/2}/[a^{1/2} + (b^{1/2} - a^{1/2})r_0]$, as in the rule.

C126. $q(y) = \frac{1}{2^{(N-1)/2}(\pi N)^{1/2}\Gamma(N/2)} \int_0^\infty dx\, x^N e^{-x^2/2} e^{-(1/2)(xy/N^{1/2} - \delta)^2}$;

$(-\infty,\infty)$, δ arbitrary, $N \in \{1,2,3, \ldots\}$.

R_x. Sample $\xi^{(N/2)-1}e^{-\xi}/\Gamma(N/2)$ for ξ on $(0,\infty)$ by C45 or C64. Set $x = (2\xi)^{1/2}$. Sample $e^{-\eta^2}/\pi^{1/2}$ for η on $(-\infty,\infty)$ by C59 or R11, and set $y = (2^{1/2}\eta + \delta)N^{1/2}/x$.

J. For the function $f(x,y)$ above, of which $q(y)$ is the y-marginal density, one finds $p(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \frac{1}{2^{(N-1)/2}(\pi N)^{1/2}\Gamma(N/2)} x^N e^{-x^2/2}$

$\cdot \int_{-\infty}^{\infty} e^{-(1/2)\left(\frac{xy}{N^{1/2}} - \delta\right)^2} dy$. For the substitution $\eta = \left(\frac{xy}{N^{1/2}} - \delta\right)\Big/2^{1/2}$,

this integral becomes $\frac{2^{1/2}N^{1/2}}{x} \int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta = \frac{2^{1/2}N^{1/2}}{x} \cdot 2 \cdot 1/2 \cdot \Gamma(1/2)$

$= \frac{2^{1/2}N^{1/2}\pi^{1/2}}{x}$. Hence $p(x) = \frac{1}{2^{(N/2)-1}\Gamma(N/2)} x^{N-1}e^{-x^2/2}$ on $(0,\infty)$, and

$p(y|x) = f(x,y)/p(x) = (1/(2\pi N)^{1/2})x \cdot e^{-(1/2)(xy/N^{1/2} - \delta)^2}$ for y on $(-\infty,\infty)$. Thus, we may sample $p(x)$ for x on $(0,\infty)$, and for this x, sample $p(y|x)$ for y on $(-\infty,\infty)$. Now for $\xi = x^2/2$ we have, for ξ on $(0,\infty)$, $p(x)\, dx = \xi^{(N/2)-1}e^{-\xi}\, d\xi/\Gamma(N/2)$, and for $\eta = \left(\frac{xy}{N^{1/2}} - \delta\right)\Big/2^{1/2}$, one finds

that $p(y|x)\, dy = e^{-\eta^2}\, d\eta/\pi^{1/2}$, with η on $(-\infty,\infty)$. Hence the rule follows from C2.

C127. $q(y) = \int_0^\infty \frac{dx\, x^{(n-4)/2}e^{-x/2H^2} \exp\{-(y - (\rho Kx/H))^2/2K^2(1 - \rho^2)x\}}{[2\pi K^2(1 - \rho^2)]^{1/2}(2H^2)^{(n-1)/2}\Gamma((n - 1)/2)}$; $(-\infty,\infty)$,

$n \in \{5,6,7, \ldots\}$, $H,K > 0$, $|\rho| < 1$.

R_x. Sample $\xi^{(n-3)/2}e^{-\xi}/\Gamma((n - 1)/2)$ for ξ on $(0,\infty)$ by C45 or C64. Set $x = 2H^2\xi$. Sample $e^{-\eta^2}/\pi^{1/2}$ for η on $(-\infty,\infty)$ by C59 or R11. Set $y = (\rho Kx)/H + \eta[2K^2(1 - \rho^2)x]^{1/2}$.

J. For the function f(x,y) above, of which q(y) is the y-marginal density one finds $p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = x^{(n-3)/2} e^{-x/2H^2} \Big/ (2H^2)^{(n-1)/2}\,\Gamma((n-1)/2)$ on $(0,\infty)$, and $p(y|x) = \frac{x^{-1/2}}{[2\pi K^2(1-\rho^2)]^{1/2}} \cdot \exp\{-(y-(\rho Kx/H))^2/2K^2(1-\rho^2)x\}$ for y on $(-\infty,\infty)$. But for $x = 2H^2\xi$, one has $p(x)\,dx = \xi^{(n-3)/2} e^{-\xi} \cdot d\xi/\Gamma((n-1)/2)$ on $(0,\infty)$, and for $y = (\rho Kx)/H + \eta[2K^2(1-\rho^2)x]^{1/2}$, one sees that $p(y|x)\,dy = e^{-\eta^2}\,d\eta/\pi^{1/2}$ on $(-\infty,\infty)$ The rule then follows from C2.

<u>C128. $q(y) = C^{-1}(e^{-ay} - e^{-by})/y$</u>; $(0,\infty)$, $0 < a < b$, $C = \ln(b/a)$.

R_x. Generate r,r' and set $y = -(\ln r')/ae^{Cr}$.

J. For the density $f(x,y) = C^{-1}e^{-xy}$, $a < x < b$, $0 < y < \infty$, one sees that the y-marginal density is $\int_a^b dx\, C^{-1}e^{-xy} = C^{-1}(e^{-ay} - e^{-by})/y = q(y)$ as given.

Following C124, we find the x-marginal density to be $p(x) = \int_0^{\infty} \left[C^{-1} \cdot e^{-xy}\right] dy = C^{-1}x^{-1}$ on (a,b), while $p(y|x) = f(x,y)/p(x) = xe^{-xy}$ for y on $(0,\infty)$. Hence, we sample p(x) for x on (a,b), setting $x = ae^{Cr}$ as in C18, and for this x, sample $p(y|x)$ for y on $(0,\infty)$, setting $y = -x^{-1}\ln r'$ as in C29. The rule follows. Note that the value of C is implied by the equation $1 = \int_a^b p(x)\,dx$.

<u>C129. $q(y) = C^{-1}\left(e^{-ay^{1/n}} - e^{-by^{1/n}}\right)\Big/y^{1/n}$</u>; $(0,\infty)$, $0 < a < b$, $n > 0$, $n \neq 1$, $C = \Gamma(n+1)(b^{1-n} - a^{1-n})/(1-n)$.

R_x. If $n < 1$, set $x = (a^{1-n} + (b^{1-n} - a^{1-n})r_0)^{1/(1-n)}$, if $n > 1$, set $x = (a^{1-n} - (a^{1-n} - b^{1-n})r_0)^{1/(1-n)}$. (Note that the <u>formulas</u> for x are identical.) Sample $\eta^{n-1}e^{-\eta}/\Gamma(n)$ for η on $(0,\infty)$ by C45, C64, or R27. Set $y = (\eta/x)^n$.

J. The density $f(x,y) = C^{-1}e^{-xy^{1/n}}$, $a < x < b$, $0 < y < \infty$, has y-marginal density $\int_a^b dx\, C^{-1}e^{-xy^{1/n}} = q(y)$ as given. Following C124, we find p(x)

$$= \int_0^\infty C^{-1}e^{-xy^{1/n}}\, dy = \Gamma(n+1)/Cx^n$$ on (a,b), so $p(y|x) = f(x,y)/p(x)$

$= x^n e^{-xy^{1/n}}/\Gamma(n+1)$ on $(0,\infty)$. Moreover, one sees that p(x) $= (1-n)x^{(1-n)-1}/(b^{1-n} - a^{1-n})$, and for $y = (\eta/x)^n$, one has $p(y|x)\, dy$ $= \eta^{n-1}e^{-\eta}\, d\eta/\Gamma(n)$, and the rule follows from C16, 21, 2. Note that 1

$$= \int_a^b p(x)\, dx = C^{-1}\Gamma(n+1) \cdot \frac{b^{-n+1} - a^{-n+1}}{-n+1}$$ determines C.

C130. $q(y) = \int_a^y dx\, f(x,y)$; (a,b), f(x,y) density for x,y on region R bounded by lines $x = a$, $y = b$, $x = y$. $(a < b)$. See Fig. C130.

R_x. Define $p(x) = \int_x^b f(x,y)\, dy$ for each x on (a,b). Sample density p(x) for x on (a,b). For this x, sample density $p(y|x) = f(x,y)/p(x)$, $x < y < b$, for y on (x,b).

J. This is an obvious modification of C124.

Note. For $(a,b) = (-\infty,\infty)$, R is the region above the line $y = x$.

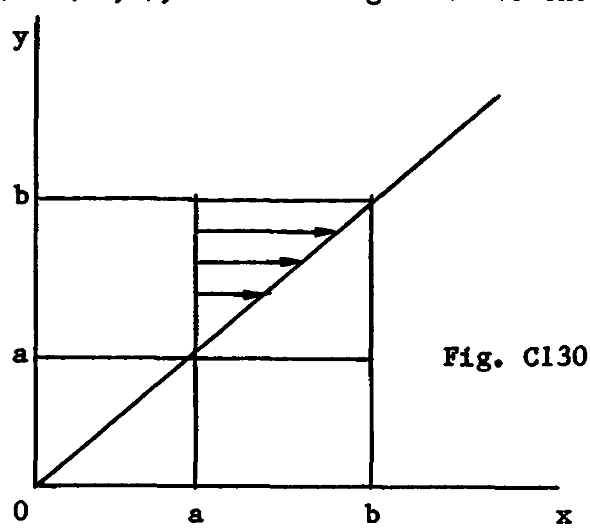


Fig. C130.

C131. $s(w) = (1/2)(g(w) + g(-w))$, $(-\infty,\infty)$, $g(z)$ density on $(-\infty,\infty)$.

R_x. Sample $g(z)$ for z on $(-\infty,\infty)$. Set $w = \pm z$ with probability 1/2.

J. According to the rule the probability of w on $(w, w + dw)$ is $(1/2)g(w)\,dw + (1/2)g(-w)\,dw$.

C132. $s(w) = (\rho/2)e^{\rho^2/2}\{\Phi(w - \rho)e^{-\rho w} + \Phi(-w - \rho)e^{\rho w}\}$; $(-\infty,\infty)$, $\rho > 0$, $\Phi(y)$

$$= (1/2\pi)^{1/2}\int_{-\infty}^{y} e^{-x^2/2}\,dx.$$

R_x. Sample $e^{-\xi^2}/\pi^{1/2}$ for ξ on $(-\infty,\infty)$ by C59 or R11. Set $z = 2^{1/2}\xi - \rho^{-1}\ell nr$, and $w = \pm z$ with probability 1/2.

J. By C131, it suffices to sample the density $g(z) = \rho e^{\rho^2/2}\Phi(z - \rho)e^{-\rho z}$ for z on $(-\infty,\infty)$, and set $w = \pm z$ as stated. But for $z = \rho + y$, one has $g(z)\,dz = \rho e^{-\rho^2/2}\Phi(y)e^{-\rho y}\,dy \equiv q(y)\,dy$, so we may sample $q(y)$ for y on $(-\infty,\infty)$ and set $z = \rho + y$. We now write $q(y)$ explicitly as $q(y) = \int_{-\infty}^{y} dx \cdot \rho e^{-\rho^2/2}e^{-x^2/2}e^{-\rho y}/(2\pi)^{1/2}$, and follow C130, i.e., we consider the density $f(x,y) = \rho e^{-\rho^2/2}e^{-x^2/2}e^{-\rho y}/(2\pi)^{1/2}$ for all points (x,y) above the line $y = x$. The y-marginal density of $f(x,y)$ is then $\int_{-\infty}^{y} dx\, f(x,y) = q(y)$ on $(-\infty,\infty)$ as above. Moreover, the x-marginal density is $p(x) = \int_{x}^{\infty} f(x,y)\,dy = e^{-\rho^2/2}e^{-x^2/2}e^{-\rho x}/(2\pi)^{1/2} = e^{-(x+\rho)^2/2}\ (2\pi)^{1/2}$ on $(-\infty,\infty)$.

Hence, $p(y|x) = f(x,y)/p(x) = \rho e^{-\rho(y-x)}$ for $x < y < \infty$. By C130, we may sample $p(x)$ for x on $(-\infty,\infty)$, and for this x, sample $p(y|x)$ for $y > x$. Now for $x = -\rho + 2^{1/2}\xi$, we have $p(x)\,dx = e^{-\xi^2}\,d\xi/\pi^{1/2}$, with ξ on $(-\infty,\infty)$ as in C59, R11. Finally, for $y = x + \rho^{-1}\eta$, we find that $p(y|x)\,dy = e^{-\eta}\,d\eta$, with η on $(0,\infty)$, as in C29, with the rule $\eta = -\ell nr$. Collecting these results, we set $z = \rho + y = \rho + (x + \rho^{-1}\eta) = \rho + (-\rho + 2^{1/2}\xi)$

$+ \rho^{-1}\eta = 2^{1/2}\xi - \rho^{-1}\ell nr$, where ξ is obtained on $(-\infty,\infty)$ from $e^{-\xi^2}/\pi^{1/2}$ by C59 or R11. The rule follows.

C133. $t(u) = \dfrac{e^{(\sigma/\phi)^2/2}}{2\phi}\left\{\Phi\left(\dfrac{u-\zeta}{\sigma} - \dfrac{\sigma}{\phi}\right)e^{-\frac{u-\zeta}{\phi}} + \Phi\left(-\dfrac{u-\zeta}{\sigma} - \dfrac{\sigma}{\phi}\right)e^{\frac{u-\zeta}{\phi}}\right\}$; $(-\infty,\infty)$, σ,

$\phi > 0$, ζ arbitrary, $\Phi(y) = (1/2\pi)^{1/2}\int_{-\infty}^{y} e^{-x^2/2}\,dx$.

R_x. Sample $s(w)$ for w on $(-\infty,\infty)$ as in C132, with $\rho \equiv \sigma/\phi$. Set $u = \zeta + \sigma w$.

J. The rule follows from C2, since for the substitution $u = \zeta + \sigma w$, one has $t(u)\,du = s(w)\,dw$ as in C132.

C134. $q(y) = C^{-1}e^{-y}\int_0^y x^{n-1}\,dx/(y-x)^n$; $(0,\infty)$, $0 < n < 1$, $C = \Gamma(n)\Gamma(1-n)$ $= \pi/\sin n\pi$. (See F4B.)

R_x. Sample $x^{n-1}e^{-x}/\Gamma(n)$ and $\eta^{(1-n)-1}e^{-\eta}/\Gamma(1-n)$ for x and η on $(0,\infty)$ by C64 or R27. Set $y = x + \eta$.

J. For the density $f(x,y) = C^{-1}e^{-y}x^{n-1}/(y-x)^n$ on the first quadrant above the line $y = x$, one has the y-marginal density $\int_0^y dx\, f(x,y) = q(y)$ as given. Following C130, we compute the x-marginal density $p(x)$

$= \int_x^\infty f(x,y)\,dy = C^{-1}x^{n-1}\int_x^\infty e^{-y}\,dy/(y-x)^n = C^{-1}x^{n-1}\int_0^\infty \eta^{-n}e^{-(x+\eta)}\,d\eta$

$= C^{-1}x^{n-1}e^{-x}\Gamma(1-n) = x^{n-1}e^{-x}/\Gamma(n)$, and the x-dependent y-density $p(y|x)$ $= f(x,y)/p(x) = (y-x)^{-n}e^{-(y-x)}/\Gamma(1-n)$. By C130, we may sample $p(x)$ for x, as in the rule, and for this x, sample $p(y|x)$ for $y > x$. But for $y = x + \eta$, one has $p(y|x)\,dy = \eta^{-n}e^{-\eta}\,d\eta/\Gamma(1-\eta)$ on $(0,\infty)$, and the rule follows from C2.

C135. $q(y) = \int_y^b dx\, f(x,y)$; (a,b), $f(x,y)$ density on region R bounded by lines $x = b$, $y = a$, $y = x(a < b!)$. See Fig. C135.

R_x. Define x-marginal density $p(x) = \int_a^x f(x,y)\, dy$, for each x on (a,b).

Sample p(x) for x on (a,b). For this x, sample the x-dependent y-density $p(y|x) = f(x,y)/p(x)$ for y on (a,x).

J. Obvious variant of C130.

<u>Note</u>. For a = 0, the region R is that bounded by the x-axis, the vertical x = b, and the line y = x.

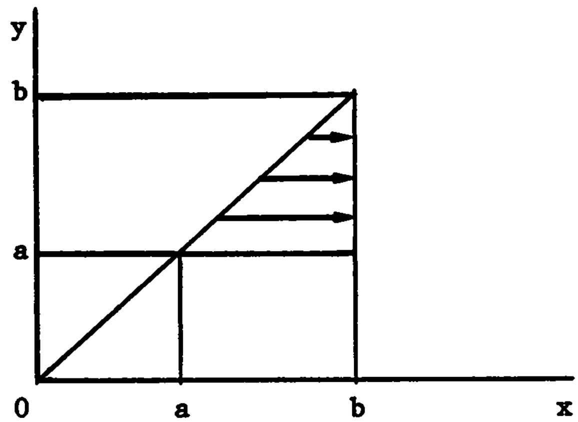


Fig. C135.

C136. $q(y) = \int_y^b dx\, t(x)/t_1$; (0,b), t(x) density on (0,b) with first moment t_1

$$= \int_0^b xt(x)\, dx.$$

R_x. Sample the density $p(x) = xt(x)/t_1$ for x on (0,b). Set $y = xr_0$.

J. This is a corollary of C135 (with a = 0). For the region R bounded by the x-axis, the vertical x = b, and the line y = x, the density f(x,y) $\equiv t(x)/t_1$ has y-marginal density q(y) as given, x-marginal density p(x) $= \int_0^x f(x,y)\, dy = (t(x)/t_1) \int_0^x dy = xt(x)/t_1$, and x-dependent y-density $p(y|x) = f(x,y)/p(x) = 1/x$ (independent of y!) Thus we sample p(x) $= xt(x)/t_1$ for x on (0,b), and for this x, sample $p(y|x) = 1/x$ for y on

(0,x). But for the latter, C1 sets $r_0 = \int_0^y dy/x = y/x$, giving $y = xr_0$, and the rule follows.

Note 1. To sample the "tail-end" density

$$q(y) = \int_y^b dx\ t(x)/t_1$$

of a density t(x) on (0,b), it suffices to be able to sample its "first moment" density $p(x) = xt(x)/t_1$ on (0,b).

Note 2. The "tail-end" density of t(x), which is really its (upper) distribution function normed by t_1, is not to be confused with its "residual" density $t(x)/T_a$ on a fixed terminal interval (a,b), where

$T_a = \int_a^b t(x)\ dx$, examples of which are given in C106, 107, 108.

C137. $q(y) = \int_y^\infty dx\ B^{n+1}x^{n-1}e^{-Bx}/\Gamma(n + 1)$; $(0,\infty)$, B, n > 0.

R_x. Sample $\xi^n e^{-\xi}/\Gamma(n + 1)$ for ξ on $(0,\infty)$ by C45, C64, or R27. Set $y = r_0(\xi/B)$.

J. This is an application of C136 to the density $t(x) = B^n x^{n-1} e^{-Bx}/\Gamma(n)$ on $(0,\infty)$, with first moment

$$t_1 = \int_0^\infty B^n x^n e^{-Bx}\ dx/\Gamma(n) = \int_0^\infty (Bx)^n e^{-Bx}\ d(Bx)/B\Gamma(n)$$

$= \Gamma(n + 1)/B\Gamma(n) = n/B$. For, $\int_y^\infty t(x)\ dx/t_1$

$$= (B/n) \int_y^\infty B^n x^{n-1} e^{-Bx}\ dx/\Gamma(n) = q(y) \text{ as given.}$$

Following C136, we find that $p(x)\ dx = xt(x)\ dx/t_1 = B^{n+1}x^n e^{-Bx}\ dx/\Gamma(n + 1) = (Bx)^n e^{-Bx}\ d(Bx)/\Gamma(n + 1) = \xi^n e^{-\xi}\ d\xi/\Gamma(n + 1)$, for $x = \xi/B$. The rule follows from C136.

C138. $q(y) = (B/n)e^{-By} \sum_{0}^{n-1} (By)^{\nu}/\nu!$; $(0,\infty)$, $B > 0$, $n \in \{1,2, \ldots\}$.

R_x. Set $y = -(r_0/B) \ln \prod_{1}^{n+1} r_i$.

J. This is the special case of C137 for n integral. For, using F3A, we see that $q(y) = \int_{y}^{\infty} dx\, B^{n+1} x^{n-1} e^{-Bx}/\Gamma(n + 1) = (B/n) \int_{y}^{\infty} d\,(Bx)(Bx)^{n-1} e^{-Bx}/\Gamma(n)$

$= (B/n) \int_{By}^{\infty} d\xi\, \xi^{n-1} e^{-\xi}/(n - 1)! = (B/n)\, e^{-By} \sum_{0}^{n-1} (By)^{\nu}/\nu! = q(y)$ as above.

The rule therefore follows from C137, since sampling $\xi^n e^{-\xi}/n!$ by C45 gives $\xi = - \ln \prod_{1}^{n+1} r_i$.

C139. $q(y) = (m + 1)(b^m - y^m)/mb^{m+1}$; $(0,b)$, $m,b > 0$.

R_x. Sample $(m + 1)x^m/b^{m+1}$ for x on $(0,b)$ by C15 or C16. Set $y = xr_0$.

J. The rule is an application of C136 to the density $t(x) = mx^{m-1}/b^m$ on $(0,b)$, with $t_1 = \int_{0}^{b} xt(x)\, dx = mb/(m + 1)$. In fact, $\int_{y}^{b} dx\, t(x)/t_1$

$= ((m + 1)/b^{m+1}) \int_{y}^{b} x^{m-1}\, dx = ((m + 1)/mb^{m+1})(b^m - y^m) = q(y)$ as given.

Moreover, $p(x) = xt(x)/t_1 = (m + 1)x^m/b^{m+1}$. The rule then follows from C136.

Note. Direct sampling by C1 leads to the equation $y^{m+1} - (m + 1)b^m y = mb^{m+1} r_0 = 0$.

C140. $q(y) = \int_a^b dx\, p(x)\, f_x(y)$; (c,d), p(x) density on (a,b), $f_x(y)$ continuous y-density on (c,d) for each value of parameter x on (a,b).

R_x. Sample p(x) for x on (a,b). For this x, sample density $f_x(y)$ for y on (c,d).

J. The function $f(x,y) \equiv p(x)f_x(y)$ is a density for x,y, since

$$\int_a^b \int_c^d f(x,y)\, dx\, dy = \int_a^b dx\, p(x) \int_c^d f_x(y)\, dy = \int_a^b dx\, p(x) = 1.$$

Moreover, the x-marginal density of this f(x,y) is $\int_c^d f(x,y)\, dy$

$= \int_c^d p(x)f_x(y)\, dy = p(x)$ itself, and $p(y|x) \equiv f(x,y)/p(x) = f_x(y)$.

Hence, C140 may be regarded as a corollary of C124.

C141. $q(y) = (m/(2\pi b)^{1/2}) \int_0^1 dx\, x^{m-(3/2)} e^{-y^2/2bx}$; $(-\infty,\infty)$, m,b > 0.

R_x. Sample the density $p(x) = mx^{m-1}$ for x on (0,1) by C15 or C16. Sample the density $e^{-v^2}/\pi^{1/2}$ for v on $(-\infty,\infty)$ by C59 or R11. Set $y = v(2bx)^{1/2}$.

J. We define $f(x,y) = (mx^{m-1})(e^{-y^2/2bx}/(2\pi bx)^{1/2})$ for $0 < x < 1$, $-\infty < y < \infty$. This is of the form $p(x)f_x(y)$ as in C140, so we sample p(x) for x on (0,1), and for this x, sample $f_x(y)$ for y on $(-\infty,\infty)$. But for y $= v(2bx)^{1/2}$, one has $f_x(y)\, dy = e^{-v^2} dv/\pi^{1/2}$ on $(-\infty,\infty)$, and the rule follows from C140 and C2.

C142. $q(y) = \{(1 + ay)e^{-ay} - (1 + by)e^{-by}\}/y^2(b - a)$; $(0,\infty)$, $0 < a < b$.

R_x. Set $y = -\ell n r_2/[a + (b - a)r_1]$.

J. For the uniform density $p(x) = 1/(b - a)$ on (a,b), and the x-dependent y-density $f_x(y) = xe^{-xy}$ on $(0,\infty)$, defined for each x on (a,b), we find that $\int_a^b dx\, p(x)f_x(y) = (b-a)^{-1}\int_a^b dx\, xe^{-yx} = (b-a)^{-1}y^{-2}\int_{ay}^{by} \xi e^{-\xi}\, d\xi$ $= q(y)$ as given. Hence, following C140, we sample $p(x) = 1/(b-a)$ for $x = a + (b-a)r_1$, as in C11, and for this x, sample $f_x(y) = xe^{-xy}$ for $y = -x^{-1}\ln r_2$, by C29. The rule follows.

Note. $q(y) > 0$, since $f(z) \equiv (1+z)e^{-z}$ is decreasing. In fact, $f'(z) = e^{-z} - (1+z)e^{-z} = -ze^{-z} < 0$. $q(y) > 0$ also follows from the integration $q(y) = (b-a)^{-1}\int_a^b dx\, xe^{-yx}$.

C143. $q(x) = k\binom{N}{k}P^{k-1}(x)p(x)[1-P(x)]^{N-k}$; (a,b), p(x) density on (a,b), $P(x) = \int_a^x p(x)\,dx$; $k \in \{1, \ldots, N\}$.

R_x. Sample p(x) independently N times for $x_1, \ldots, x_N$. Order these x_i as $x_1' \leq x_2' \leq \cdots \leq x_N'$. Set $x = x_k'$.

J. q(x) is the density for x_k', the k-th largest component of the vector $(x_1, \ldots, x_N)$, the x_i being independent, each with density p(x). For, the corresponding distribution function $Q(x) = \int_a^x q(x)\,dx = P\{x_k' \leq x\}$ is the probability that at least k of the x_i are $\leq x$. Hence $Q(x) = \binom{N}{k}P^k(x)[1-P(x)]^{N-k} + \binom{N}{k+1}P^{k+1}(x)[1-P(x)]^{N-k-1} + \cdots + \binom{N}{N-1} \cdot P^{N-1}(x)[1-P(x)] + \binom{N}{N}P^N(x)$, and one finds that $q(x) = Q'(x) = \binom{N}{k}kP^{k-1}(x)p(x)[1-P(x)]^{N-k}$. (The derivative sum telescopes, leaving only the first term.)

Note. In densities of this form, the above rule is feasible for moderate N, and may compare favorably with more direct methods.

C144. A. $q(x) = Np(x)[1-P(x)]^{N-1}$; (a,b),

B. $q(x) = NP^{N-1}(x)p(x)$; (a,b),

C. $q(x) = ((2M+1)!/M!M!)p(x)[P(x)(1-P(x))]^M$; (a,b).

In A,B,C, $p(x)$ is a density on (a,b), and $P(x) = \int_a^x p(x)\,dx$.

R_x. Sample $p(x)$ for $x_1, \ldots, x_N$, where $N = 2M + 1$ in case C. For A, set $x = \min\{x_i\}$; for B, set $x = \max\{x_i\}$; for C, where $N = 2M + 1$ is odd, order the $x_1, \ldots, x_{2M+1}$ as $x_1' \le \cdots \le x_{2M+1}'$ and set $x = x_{M+1}'$ (the middle x_i').

J. One sees from C143 that the above densities are respectively those for the least $(k = 1)$, the greatest $(k = N)$, and for $N = 2M + 1$, the middle $(k = M + 1)$ component in size in the sequence of x_i.

C145. $q(x) = k\binom{N}{k}(x - a)^{k-1}(b - x)^{N-k}/(b - a)^N$; (a,b), $k \in \{1, \ldots, N\}$.

R_x1. Generate $r_1, \ldots, r_N$. Order as $r_1' \le \cdots \le r_N'$. Set $x = a + (b - a)r_k'$.

J1. For the uniform density $p(x) = 1/(b - a)$ on (a,b), one has $P(x)$

$$= \int_a^x p(x)\,dx = (x - a)/(b - a), \text{ and } 1 - P(x) = (b - x)/(b - a).$$

Substitution in C143 gives the above $q(x)$. Hence we should sample $p(x)$ for $x_1 = a + (b - a)r_1, \ldots, x_N = a + (b - a)r_N$, where the order of the x_i is that of the corresponding r_i. The rule follows.

R_x. Define $m = k$, $n = N - k + 1$, and sample the density $(x - a)^{m-1} \cdot (b - x)^{n-1}/(b - a)^{m+n-1}B(m,n)$ for x on (a,b) as in C76.

J2. This is an obvious alternative.

C146. $q(x) = k\binom{N}{k}x^{k-1}(1 - x)^{N-k}$; $(0,1)$, $k \in \{1, \ldots, N\}$.

R_x. Generate $r_1, \ldots, r_N$. Order as $r_1' \le \cdots \le r_N'$. Set $x = r_k'$.

J1. Case $a = 0$, $b = 1$ of C145.

R_x. Define $m = k$, $n = N - k + 1$. Sample density $x^{m-1}(1 - x)^{n-1}/B(m,n)$ for x on $(0,1)$ by C75.

J2. An obvious alternative.

Note 1. The method of R_x1 provides a useful test for the randomness of machine generated "random numbers."

Note 2. For $k = N$, the rule R_x1 samples $q(x) = Nx^{N-1}$ for x on $(0,1)$ by setting $x = \max\{r_1, \ldots, r_N\}$. The direct method of C1 would set $x = r_0^{1/N}$. See C15,16.

C147. $q(x) = k\binom{N}{k}e^{-x}\exp(-ke^{-x})[1 - \exp(-e^{-x})]^{N-k}$; $(-\infty,\infty)$, $k \in \left\{1, \ldots, N\right\}$.

R_x. Generate $r_1, \ldots, r_N$. Order as $r_1' \leq \cdots \leq r_N'$. Set $x = -\ell n(-\ell n r_k')$.

J. For the density $p(x) = \exp(-x - e^{-x})$ on $(-\infty,\infty)$ of C43, one has $P(x)$

$$= \int_{-\infty}^{x} p(x)\, dx = \exp(-e^{-x}). \quad \text{(Let } x = -\ell n\xi.) \text{ Substitution in C143 gives}$$

the above q(x). To sample p(x), we set $x = -\ell n(-\ell n r)$ as in C43. Since the function $-\ell n(-\ell n r)$ is increasing, the rule follows.

Note. For $k = N$, $q(x)\, dx = Ne^{-x} \exp(-Ne^{-x})\, dx = Ne^{-N\eta}(-d\eta) = e^{-\zeta}(-d\zeta)$ under the substitutions $x = -\ell n\eta$, $\eta = \zeta/N$. By C2 and C29, we could set $x = -\ell n(-N^{-1}\ell n r_0)$.

C148. $q(x) = k\binom{N}{k}e^{-(N-k+1)x}/(1 + e^{-x})^{N+1}$; $(-\infty,\infty)$, $k \in \{1, \ldots, N\}$.

R_x1. Generate $r_1, \ldots, r_N$. Order as $r_1' \leq \cdots \leq r_N'$. Set $x = \ell n(r_k'/(1 - r_k'))$.

J1. For the density $p(x) = e^{-x}/(1 + e^{-x})^2$ on $(-\infty,\infty)$ of C120, one has $P(x)$

$$= \int_{-\infty}^{x} p(x)\, dx = 1/(1 + e^{-x}). \quad \text{Substitution in C143 gives the above } q(x).$$

The rule then follows from C120, since the function $\ell n(r/(1 - r)) = \ell n(1/(r^{-1} - 1))$ is increasing.

R_x2. Define $m = N - k + 1$, $n = k$, $\rho = \sigma = 1$. Sample $e^{-mx}/(1 + e^{-x})^{m+n}B(m,n)$ for x as in C79.

J2. An obvious alternative.

C149. $$q(x) = \frac{k\binom{N}{k}\left[(1/2) + (1/\pi)\arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{k-1}\left[(1/2) - (1/\pi)\arctan\left(\frac{x-\theta}{\lambda}\right)\right]^{N-k}}{\pi\lambda\left[1 + \left(\frac{x-\theta}{\lambda}\right)^2\right]};$$

$(-\infty,\infty)$, $\lambda > 0$, θ arbitrary, $k \in \{1, \ldots, N\}$.

R_x. Sample the density p(x) of C99 for $x_1, \ldots, x_N$. Order as $x_1' \leq \cdots \leq x_N'$. Set $x = x_k'$

J. For $p(x) = 1/\pi\lambda[1 + ((x - \theta)/\lambda)^2]$ in C99, one has $P(x) = \int_{-\infty}^{x} p(x)\, dx$

$= (1/2) + (1/\pi) \arctan((x - \theta)/\lambda)$. Substitution in C143 gives the above q(x). The rule follows from C143.

Note. If R_x1 of C99 is used, one may generate $r_1, \ldots, r_N$, order as $r'_1 \le \ldots \le r'_N$, and set $x = \theta + \lambda \tan(2r'_k - 1)(\pi/2)$.

C150. $q(x) = k\binom{N}{k} abx^{b-1} e^{-ax^b(N-k+1)} \left[1 - e^{-ax^b}\right]^{k-1}$; $(0,\infty)$, $a,b > 0$, $k \in \{1, \ldots, N\}$.

R_x. Generate $r_1, \ldots, r_N$. Order as $r'_1 \ge r'_2 \ge \ldots \ge r'_N$ (Sic!). Set $x = \exp[b^{-1} \ln(-a^{-1} \ln r'_k)]$.

J. For the Weibull density $p(x) = abx^{b-1}e^{-ax^b}$ on $(0,\infty)$ in C38, one has

$$P(x) = \int_0^x p(x)\, dx = 1 - e^{-ax^b},$$

and substitution in C143 gives the above $q(x)$. The rule follows from C38, since the function $x = \exp[b^{-1} \ln(-a^{-1} \ln r)]$ is decreasing, with

$$r'_1 \ge \ldots \ge r'_k \ge \ldots \ge r'_N$$

corresponding to $x'_1 \le \ldots \le x'_k \le \ldots \le x'_N$.

C151. $q(x) = k\binom{N}{k}\left(1 - e^{-x}\right)^{k-1} e^{-(N-k+1)x}$; $(0,\infty)$, $k \in \{1, \ldots, N\}$.

R_x1. Generate $r_1, \ldots, r_N$. Order as $r'_1 \ge \ldots \ge r'_N$. Set $x = -\ln r'_k$.

J1. Case $a = b = 1$ of C150.

R_x2. Define $m = N - k + 1$, $n = k$, $\rho = \sigma = 1$. Sample the density $e(x)$ for x on $(0,\infty)$ by C80.

J2. Obvious alternative.

C152. $q(x) = k\binom{N}{k}(\Gamma_x(n))^{k-1}[x^{n-1}e^{-x}/\Gamma(n)][1 - \Gamma_x(n)]^{N-k}$; $(0,\infty)$, $n > 0$, $\Gamma_x(n)$

$$\equiv \int_0^x x^{n-1}e^{-x}\, dx/\Gamma(n),\ k \in \{1, \ldots, N\}.$$

R_x. Sample density $p(x) = x^{n-1}e^{-x}/\Gamma(n)$ N times for $x_1, \ldots, x_N$ on $(0,\infty)$ by C45, C64, or R27. Order as $x'_1 \le \ldots \le x'_N$. Set $x = x'_k$.

J. Since $p(x)$ has the distribution function $P(x) = \int_0^x x^{n-1}e^{-x}\, dx/\Gamma(n)$ $= \Gamma_x(n)$ as defined, substitution in C143 gives the above $q(x)$, and the rule follows from C143.

Note. For an integer $n = 1,2,3, \ldots$, one has $\Gamma_x(n) = \int_0^x x^{n-1}e^{-x}\,dx/(n - 1)! = 1 - \int_x^\infty x^{n-1}e^{-x}\,dx/(n-1)! = 1 - e^{-x}\sum_0^{n-1} x^\nu/\nu!$ from F3A.

C153. $q(x) = k\binom{N}{k}(m/x)(\beta/x)^{m(N-k+1)}[1 - (\beta/x)^m]^{k-1}$, (β,∞), $m,\beta > 0$, $k \in \{1, \ldots, N\}$.

R_x. Sample the density $m\beta^m x^{-m-1}$ for $x_1, \ldots, x_N$ on (β,∞) by C20 or C21. Order as $x_1' \leq \ldots \leq x_N'$. Set $x = x_k'$.

J. For the density $p(x) = m\beta^m x^{-m-1}$, one has $P(x) = \int_\beta^x p(x)\,dx = 1 - (\beta/x)^m$, and substitution in C143 gives the above $q(x)$. The rule therefore follows from C143.

Note. If the formula $x = \beta/r^{1/m}$ of C21 is used to sample $p(x)$, one may generate $r_1, \ldots, r_N$, order as $r_1' \geq \ldots \geq r_N'$, and set $x = \beta/(r_k')^{1/m}$.

C154. $p(t) = (\lambda/2\pi)^{1/2}t^{-3/2}\exp\{-\lambda(t - \mu)^2/2\mu^2 t\}$; $(0,\infty)$, $\lambda,\mu > 0$.

R_x. Define $\phi = \lambda/2\mu$, and sample $q(x)$ for x as in R25. Set $t = \mu x$.

J. For $t = \mu x$, and $\phi = \lambda/2\mu$, one sees that $p(t)\,dt = q(x)\,dx$, where $q(x)$ is Wald's density in R25.

C155. $p(t) = (d/(2\pi\beta)^{1/2})t^{-3/2}\exp\{-(d - vt)^2/2\beta t\}$; $(0,\infty)$, $\beta,d,v > 0$. (Brownian motion.)

R_x. Sample $p(t)$ for t as in C154, with the parameter values $\lambda = d^2/\beta, \mu = d/v$.

J. Under this identification of parameters, C155 is a case of C154.

C156. $f(x,y) = C^{-1}x^{m-1}y^{n-1}F(x + y)$; Region: $\{(x,y);\ x,y > 0,\ x + y < a\}$, a fixed, $0 < a \leq \infty$, $m,n > 0$, $C = A \cdot B(m,n)$, where $A \equiv \int_0^a u^{m+n-1}F(u)\,du$.

R_x. Sample density $A^{-1}u^{m+n-1}F(u)$ for u on $(0,a)$. Sample density $v^{m-1}(1 - v)^{n-1}/B(m,n)$ for v on $(0,1)$ by C75 or R28. Set $x = uv$, $y = u(1 - v)$.

J. Since the Jacobian of the preceding transformation is $-u$, one has $f(x,y)\,dx\,dy = A^{-1}u^{m+n-1}F(u)\,du\,B^{-1}(m,n)v^{m-1}(1 - v)^{n-1}\,dv$ on $(0,a)$

X (0,1), and the rule follows from C2.

Note 1. The inverse of the transformation, $u = x + y$, $v = x/(x + y)$, shows that the (x,y) region is mapped into the rectangular (u,v) region (0,a) X (0,1), downward diagonals going into upward verticals.

Note 2. For the function $F(z) = e^{-z}$ and $a = \infty$, see C75.

C157. $f(x,y) = C^{-1}x^{m-1}y^{n-1}/(1 - x - y)^n$, Region: $\{(x,y),\ x,y > 0,\ x + y \le 1\}$, $m > 0$, $0 < n < 1$, $C = \pi/m \sin n\pi$.

R_x. Sample $u^{m+n-1}(1 - u)^{(1-n)-1}/B(m + n,\ 1 - n)$, and $v^{m-1}(1 - v)^{n-1}/B(m,n)$ for u,v on (0,1) by C75 or R28. Set $x = uv$, $y = u(1 - v)$.

J. Case $a = 1$, $F(z) = 1/(1 - z)^n$ of C156.

Note. $$A = \int_0^1 u^{m+n-1}(1 - u)^{-n}\, du = B(m + n, 1 - n),$$

$$C \equiv AB(m,n) = \frac{\Gamma(m + n)\Gamma(1 - n)}{\Gamma(m + 1)} \cdot \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)} = \Gamma(n)\Gamma(1 - n)/m = \pi/m \sin n\pi.$$

See F4B.

C158. $p(x_1,x_2) = (2\pi\sigma_1\sigma_2 R)^{-1}e^{-Q/2R^2}$, $Q = \left(\frac{x_1 - \mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1 - \mu_1}{\sigma_1}\right)\left(\frac{x_2 - \mu_2}{\sigma_2}\right) + \left(\frac{x_2 - \mu_2}{\sigma_2}\right)^2$; x_1,x_2 on $(-\infty,\infty)$, $\sigma_1,\sigma_2 > 0$, μ_1,μ_2 arbitrary, $R^2 = 1 - \rho^2$, $|\rho| < 1$.

R_x. Sample $e^{-y^2}/\pi^{1/2}$ for y_1,y_2 on $(-\infty,\infty)$ by C59 or R11. Set

$$x_1 = \mu_1 + 2^{1/2}\sigma_1(Ry_1 + \rho y_2),$$

$$x_2 = \mu_2 + 2^{1/2}\sigma_2 y_2.$$

J. Under the preceding transformation, with Jacobian $2\sigma_1\sigma_2 R$, one finds that

$$p(x_1,x_2)\, dx_1\, dx_2 = (2\pi\sigma_1\sigma_2 R)^{-1}e^{-\left(y_1^2 + y_2^2\right)}(2\sigma_1\sigma_2 R)\, dy_1\, dy_2$$

$$= \pi^{-1/2}e^{-y_1^2}\, dy_1 \cdot \pi^{-1/2}e^{-y_2^2}\, dy_2,$$ and the rule follows from C2.

C159. $p(x_1,x_2) = (2\pi R)^{-1}e^{-Q/2R^2}$, $Q = x_1^2 - 2\rho x_1x_2 + x_2^2$, x_1,x_2 on $(-\infty,\infty)$, $R^2 = 1 - \rho^2$, $|\rho| < 1$.

R_x. Set $x_1 = 2^{1/2}(Ry_1 + \rho y_2)$, $x_2 = 2^{1/2}y_2$.

J. Case $\mu_1 = \mu_2 = 0$, $\sigma_1 = \sigma_2 = 1$ of C158.

C160. $p(x_1, \ldots, x_n) = C^{-1}e^{-Q}$, $Q = \sum x_i a_{ij} x_j$; x_j on $(-\infty,\infty)$, $j = 1, \ldots, n$,

$n \times n$ matrix $A = [a_{ij}]$ symmetric, positive definite.

R_x. Construct $n \times n$ matrix S such that $S^T AS = I$. (See Note.) Sample $e^{-y^2}/\pi^{1/2}$ independently n times for $y_1, \ldots, y_n$ on $(-\infty,\infty)$ by C59 or R11. Define $x_1, \ldots, x_n$ by the linear transformation

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = S \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = SY.$$

J. In column vector notation, we have the transformation $X = SY$, and hence

$$Q = X^T AX = (SY)^T A(SY) = Y^T(S^T AS)Y = Y^T IY = Y^T Y, \text{ i.e., } Q = \sum x_i a_{ij} x_j$$

$= \sum y_i^2$. The Jacobian of the transformation being det S, we see that

$$p(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = C^{-1} \exp -\left(\sum y_i^2\right) \cdot |\det S|$$

$$\cdot\, dy_1 \ldots dy_n, = \pi^{-n/2} e^{-\sum y_i^2}\, dy_1 \ldots dy_n = \left(e^{-y_1^2}\, dy_1/\pi^{1/2}\right) \ldots$$

$\left(e^{-y_n^2}\, dy_n/\pi^{1/2}\right)$, where $C^{-1}\, |\det S| = \pi^{-n/2}$ necessarily. The rule then follows from C2.

Note. The matrix S may be obtained by the Gram-Schmidt process [17]. Without going into its machinery, we remark here that it is a definite algorithm for constructing, from any n linearly independent vectors, an equivalent set (spanning the same space) which are orthonormal with respect to an arbitrary given inner product. If we define in E^n the inner product $(X,Y) = X^T AY$, then the Gram-Schmidt algorithm, applied to the linearly independent "1-spot" vectors $\delta_1, \ldots, \delta_n$, produces a set of vectors $S_1, \ldots, S_n$ which are orthonormal with respect to this inner product (X,Y), and we need only define the matrix S as $S = [S_1, \ldots, S_n]$ with the vectors S_i as its columns. For then $S = [S_1, \ldots, S_n] = I \cdot [S_1, \ldots, S_n] = [\delta_1, \ldots, \delta_n][S_1, \ldots, S_n]$, and hence $\delta_{ij} = (S_i, S_j)$

$$= \left(\sum_k \delta_k s_{ki}, \sum_\ell \delta_\ell s_{\ell j}\right) = \sum_{k,\ell} s_{ki}(\delta_k, \delta_\ell) s_{\ell j} = \sum_{k,\ell} s_{ki} a_{k\ell} s_{\ell j}, \text{ or in}$$

matrix form, $I = S^T AS$.

C161. $P(\alpha', \alpha)$; Compton scattering, Klein-Nishina cross section.

R_x. The following is a complete procedure for obtaining, from an arbitrary incident photon energy $\alpha = E(\text{MeV})/.511 \geq .002$, the value of the scattered photon energy $E' = .511\ \alpha'$ MeV, where $\alpha' = \alpha x$, and the resulting deflection cosine $\mu = \cos\theta = 1 + (1/\alpha) - (1/\alpha')$. Set $\alpha_0 = 202$.

1. $\eta = 1 + 2\alpha$
2. $\xi = 1/\eta$
3. $N = \ln\eta$
4. $\alpha \geq \alpha_0 \rightarrow (5)$, $\alpha < \alpha_0 \rightarrow (11)$.
5. $T = 1 - \xi^2$
6. $G_1 = N + (T/2)$
7. Generate r, r'
8. $G_1 r' < N \rightarrow (9)$, $G_1 r' \geq N \rightarrow (10)$.
9. $x = \exp(-Nr)$, EXIT.
10. $x = (1 - rT)^{1/2}$, EXIT.
11. $\beta = 1/\alpha$
12. $\phi = \phi(\alpha)$. See TABLE.
13. $x_0 = \xi + (1 - \xi)\phi$
14. $M = \ln x_0$
15. $K_1 = 1 - x_0$
16. $K_2 = 1/x_0$
17. $K_3 = 1 - 2\beta(1 + \beta)$
18. $F_0 = K_1\{(1/2)(1 + x_0) + \beta^2(\eta + K_2) - MK_3\}$
19. $G = 2\alpha(1 + \alpha)\xi^2 + 4\beta + NK_3$
20. $J_0 = F_0/G$
21. Generate r.

22. $r < J_0 \to (23)$, $r \geq J_0 \to (29)$.

23. $R = r/J_0$

24. $f_0 = x_0 + K_2 + \beta^2(1 - K_2)(\eta - K_2)$

25. $A_0 = - F_0/2$

26. $B_0 = F_0 + (F_0/f_0) - 3K_1$.

27. $C_0 = A_0 - (F_0/f_0) + 2K_1$

28. $x = 1 + R\{A_0 + R(B_0 + RC_0)\}$, EXIT

29. $\Lambda_0 = (M + N)/(1 - J_0)$

30. $x = x_0 \exp\{-\Lambda_0(r - J_0)\}$, EXIT

TABLE I

$.002 \leq \alpha < .962$	$\phi = .25$	$\lvert\varepsilon\rvert = .0211$
$.962 \leq \alpha < 1.642$	$\phi = .20$	$\lvert\varepsilon\rvert = .0218$
$1.642 \leq \alpha < 2.002$	$\phi = .17$	$\lvert\varepsilon\rvert = .0218$
$2.002 \leq \alpha < 10$	$\phi = .15$	$\lvert\varepsilon\rvert = .0213$
$10 \leq \alpha < 52$	$\phi = .25$	$\lvert\varepsilon\rvert = .0177$
$52 \leq \alpha < 202$	$\phi = .25$	$\lvert\varepsilon\rvert = .0194$

The tabulated $|\varepsilon|$ is the maximum relative error in x on the corresponding range.

J. The rule is based on an accurate fit for the inverse of the Klein-Nishina distribution. For details see [11,16,14]. This is cited in [15] and is an improvement on the method used in the earlier version [10].

R-INDEX

Rejection Techniques

R1.	$\cos\theta, \sin\theta, \tan\theta$; $p(\theta) = 2/\pi;\ (0,\pi/2)$.	Uniform direction, quadrant I.
R2.	$\cos\theta, \sin\theta, \tan\theta$; $p(\theta) = 1/\pi;\ (-\pi/2,\pi/2)$.	Uniform direction, quadrants I, IV.
R3.	$\cos\phi, \sin\phi$; $p(\phi) = 1/2\pi;\ (0,2\pi)$.	Uniform direction in plane, point on unit circle.
R4.	$(a^2 - y^2)^{1/2}$.	Sampling torus, uniform in volume.
R5.	$\Omega = (\omega_1,\omega_2,\omega_3)$.	Uniform direction in 3-space, point on unit sphere.
R6.	$\Omega = (\omega_1, \ldots, \omega_N)$.	Uniform direction in N-space, point on unit sphere $\lvert\Omega\rvert = 1$.
R7.	$p_1(x)\{P_2(h(x)) - P_2((g(x))\}$.	Density x distribution.
R8.	$e^{-x^2/2};\ (0,\infty)$.	Half normal.
R9.	$e^{-y^2/2};\ (-\infty,\infty)$.	Normal.
R10.	$e^{-\nu^2};\ (0,\infty)$.	Half Gaussian.
R11.	$e^{-x^2};\ (-\infty,\infty)$.	Gaussian.
R12.	$e^{-az} \sinh (bz)^{1/2}$.	Fission energy spectrum, Watt spectrum.
R13.	$p_1(x)h(x)$.	Density x bounded function.
R14.	$(b - a)^{-1}p(x)/\hat{p}$.	Uniform x bounded density.
R15.	$F^{-1}f(x)$	Uniform x bounded function.
R16.	$K - B\cos^2\phi$.	Polarized Compton Scattering.
R17.	$K - S^2(Q\cos 2\phi + U\sin 2\phi)$.	Polarized Compton Scattering.
R18.	$\begin{cases} h(x - a)/(b - a) \\ h(c - x)/(c - b) \end{cases}$.	General triangular.
R19.	$\sin^2 x/x^2$.	Quasi-periodic.

R20. $\sum \alpha_j p_j(x) h_j(x)$. — Sum of products, Butcher.

R21. $\nu^{n-1}/(\Lambda^{-1}e^{\nu} + 1)$. — NR non-degenerate electron gas energy.

R22. $y^{1/2}/(e^{y-\eta} + 1)$. — Fermi-Dirac.

R23. $x^{n-1}e^{-\xi(x^2+1)^{1/2}}$. — R extreme non-degenerate electron gas momentum, Maxwell-Juttner.

R24. $x^{n-1}\Big/\left(\Lambda^{-1}e^{a(x^2+1)^{1/2}} + 1\right)$. — R non-degenerate electron gas momentum.

R25. $x^{-3/2} \exp\{-\phi(x - 1)^2/x\}$. — Wald. [22, v.2; p. 138]

R26. $(1 - R^2)^{(T-1)/2}/(1 + \rho^2 - 2\rho R)^{T/2}$. — Leipnik, circular correlation. [22, v.3; p. 240]

R27. $x^{m-1}e^{-x}$, $m > 0$. — General Γ-type. [22, v.3; p. 39]

R28. $v^{m-1}(1 - v)^{n-1}$, — General B-type. [22, v.3; p. 39]

$z^{m-1}/(1 + z)^{m+n}$,

$\sin^{2m-1}\theta \cos^{2n-1} \theta$; $m,n > 0$.

R29. $z^{2m-1}e^{-z^2}$, $m > 0$. — General Gauss type.

R30. $P(\alpha'/\alpha,\theta)$. — Polarized Klein-Nishina.

Rejection Techniques

Note. In all cases, the process is iterated until the stated condition for acceptance is satisfied.

R1. $\cos\theta$, $\sin\theta$, $\tan\theta$, for $p(\theta) = 2/\pi$; $(0,\pi/2)$.

R_x. If $S = \hat{r}_1^2 + \hat{r}_2^2 \le 1$, set $\cos\theta = \hat{r}_1/S^{1/2}$, $\sin\theta = \hat{r}_2/S^{1/2}$, $\tan\theta = \hat{r}_2/\hat{r}_1$.

J. The accepted points $(\hat{r}_1,\hat{r}_2)$ constitute a sampling of the unit disk in the first quadrant which is uniform in area. The angles θ so determined are therefore uniform on $(0,\pi/2)$.

Note. In this and other such cases, there is always the alternative of sampling the angle uniformly (one random number) and computing the values of the desired functions. Machine times should be compared.

R2. $\cos\theta$, $\sin\theta$, $\tan\theta$ for $p(\theta) = 1/\pi$; $(-\pi/2, \pi/2)$.

R_x. If $S = x^2 + y^2 \leq 1$, where $x = r_1$, $y = 2r_2 - 1$, set $\cos\theta = x/S^{1/2}$, $\sin\theta = y/S^{1/2}$, $\tan\theta = y/x$ $(x \neq 0)$.

J. See R1J.

R3. $\cos\phi$, $\sin\phi$ for $p(\phi) = 1/2\pi$; $(0, 2\pi)$.

R_x1. If $S = x^2 + y^2 \leq 1$, where $x = 2r_1 - 1$, $y = r_2$, set $\cos\phi = (x^2 - y^2)/S$, $\sin\phi = 2xy/S$. (von Neumann.)

J1. For accepted (x,y), $\cos\theta = x/S^{1/2}$, $\sin\theta = y/S^{1/2}$ are functions of an angle θ uniform on $(0,\pi)$. Hence $\phi = 2\theta$ is uniform on $(0, 2\pi)$, and $\cos\phi = \cos 2\theta = \cos^2\theta - \sin^2\theta = (x^2 - y^2)/S$, $\sin\phi = \sin 2\theta = 2\sin\theta\cos\theta = 2xy/S$. (No square roots required.)

R_x2. Use R1 to obtain $\cos\theta$, $\sin\theta$ for θ uniform on $(0, \pi/2)$. For $\cos\phi$, $\sin\phi$, change sign of each independently with probability 1/2.

J2. Obvious.

R_x3. If $S = x^2 + y^2 \leq 1$, where $x = 2r_1 - 1$, $y = 2r_2 - 1$, set $\cos\phi = x/S^{1/2}$, $\sin\theta = y/S^{1/2}$.

J3. See R1J.

<u>Note</u>. $\Omega = (\cos\phi, \sin\phi)$ is a uniform direction in E^2.

R4. $p(y) = (2/\pi a^2)(a^2 - y^2)^{1/2}$; $(-a, a)$, $a > 0$.

R_x. If $(2r_1 - 1)^2 + (2r_2 - 1)^2 \leq 1$, set $y = a(2r_2 - 1)$.

J. For $y = a\eta$, one has $p(y)\,dy = (2/\pi)(1 - \eta^2)^{1/2}\,d\eta \equiv q(\eta)\,d\eta$, so we may sample $q(\eta)$ for η on $(-1,1)$ and set $y = a\eta$. Since the accepted points (ξ,η), $\xi = 2r_1 - 1$, $\eta = 2r_2 - 1$ sample the unit disk uniformly in area, they have the density function $f(\xi,\eta) = 1/\pi$, with η-marginal density

$$2\int_0^{(1-\eta^2)^{1/2}} d\xi/\pi = (2/\pi)(1 - \eta^2)^{1/2} = q(\eta)$$

as defined, and the rule follows.

<u>Note</u>. The above density $p(y)$ arises naturally in the problem of sampling a toroidal solid (anchor ring) uniformly in volume. Suppose the latter generated by revolving about the y-axis a circular area of radius a, with center at $(b,0)$ $b > a$, on the x-axis. Our object is to sample the circular area for points (x,y) in such a way that they produce by rotation points (ξ,η,ζ),

$\xi = x \cos \phi$

$\eta = y$

$\zeta = x \sin \phi$

(ϕ uniform on $(0,2\pi)$) which are uniform in volume within the ring. The volume of the toroidal ring up to height y is given by

$$V(y) = \int_{-a}^{y} \pi(x_2^2 - x_1^2)\, dy$$

where $x_1 = b - (a^2 - y^2)^{1/2}$, $x_2 = b + (a^2 - y^2)^{1/2}$. Thus $V(y) = 4\pi b \int_{-a}^{y} (a^2 - y^2)^{1/2}\, dy$, the total voluem being $V(a) = 4\pi b \cdot \int_{-a}^{a} (a^2 - y^2)^{1/2}\, dy = 2\pi^2 a^2 b$. The probability distribution function for y is therefore $P(y) = V(y)/V(a) = (2/\pi a^2) \int_{-a}^{y} (a^2 - y^2)^{1/2}\, dy$, with density $p(y) = dP/dy = (2/\pi a^2)(a^2 - y^2)^{1/2}$, as in R4. For a value of y, drawn from this density as in the rule, the corresponding value of x should be uniformly distributed in area in the annulus generated by the points (x_1,y) and (x_2,y), i.e., we should set

$$r_3 = (x^2 - x_1^2)/(x_2^2 - x_1^2),$$

and solve for x. (See C12, Note.) Using the above values of x_1,x_2 the result is

$$x = \{b^2 + a^2 - y^2 + 2b(a^2 - y^2)^{1/2}(2r_3 - 1)\}^{1/2},$$

where y is obtained from the rule.

R5. $\Omega = (\omega_1,\omega_2,\omega_3)$, uniform direction in E^3, point uniform in area on the sphere $|\Omega| = (\omega_1^2 + \omega_2^2 + \omega_3^2)^{1/2} = 1$.

R_x1. Obtain $\cos\phi$, $\sin\phi$ for ϕ uniform on $(0,2\pi)$, as in R3. Set $\cos\theta = 2r_3 - 1$, $\sin\theta = +(1-\cos^2\theta)^{1/2}$, and $\Omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

J1. For spherical coordinates, $p(\theta,\phi)\,d\theta\,d\phi = \sin\theta\,d\theta\,d\phi/4\pi = p_1(\theta)\,d\theta \cdot p_2(\phi)\,d\phi$, where $p_1(\theta)\,d\theta = (1/2)\sin\theta\,d\theta = -(1/2)\,d(\cos\theta) = -(1/2)\,d\mu$; $-1 \le \mu \le 1$, $p_2(\phi)\,d\phi = d\phi/2\pi$, $0 \le \phi \le 2\pi$, where μ and ϕ are uniform. The rule follows from C6, C2.

R_x2. Obtain S, $\cos\phi$, $\sin\phi$ from R3, R_x3. Set $\cos\theta = 2S - 1$, $\sin\theta = +(1-\cos^2\theta)^{1/2}$, and Ω as in R_x1. (random number r_3 avoided.)

J2. The accepted points (x,y) in R3, R_x3 sample the unit disk, uniformly in area, having density $dx\,dy/\pi$. The equivalent ρ,θ density is $\rho\,d\rho\,d\theta/\pi$, with marginal ρ-density $2\rho\,d\rho$. Under the latter density the function $S = \rho^2$ has density

$$d/dS \int_{\rho^2 \ge S} 2\rho\,d\rho = d/dS \int_0^{\sqrt{S}} 2\rho\,d\rho = (1/2)\,S^{-1/2} \cdot 2S^{1/2} = 1.$$

Hence, S is itself distributed uniformly on $(0,1)$ and may be used in place of r_3 in R_x1 above.

<u>R6. $\Omega = (\omega_1, \ldots, \omega_N)$</u> uniform direction in E^N, uniformly distributed point on unit N-sphere $|\Omega| = 1$.

R_x. If $S = \nu_1^2 + \ldots + \nu_N^2 \le 1$, where $\nu_i = 2r_i - 1$, set $\omega_i = \nu_i/S^{1/2}$.

J. Accepted points $(\nu_1, \ldots, \nu_N)$ are uniformly distributed in volume in the unit N-sphere, and $(\omega_1, \ldots, \omega_N)$ is the projection of $(\nu_1, \ldots, \nu_N)$ on the unit N-sphere surface.

<u>Note 1</u>. For $N = 2$, this is the method of R3, R_x3. For $N = 3$, it provides an alternative to R5.

<u>Note 2</u>. Unfortunately, acceptance is poor for large N. In fact, (see F8), $E \equiv V(1)/2^N = \pi^{N/2}/2^{N-1}N\Gamma(N/2) \to 0$, the ratio becoming less than 1/2 for $N > 3$. See C90 for an alternative.

N	E
1	1
2	$\pi/4$
3	$\pi/6$
4	$\pi^2/32$
5	$\pi^2/60$

R7. $p(x) = A^{-1}p_1(x)\{P_2(h(x)) - P_2(g(x))\}$; (a,b), $p_1(x)$ density for x on (a,b), $p_2(y)$ density for y on (c,d), $P_2(y) = \int_c^y p_2(y)\,dy$, $c \le g(x) \le h(x) \le d$.

R_x. Sample $p_1(x)$ for x on (a,b), and $p_2(y)$ for y on (c,d). Accept x if $g(x) \le y \le h(x)$.

J. Since $\int_{\{g(x) \le y \le h(x)\}} p_1(x)\,dx\,p_2(y)\,dy = \int_a^b p_1(x)\,dx \int_{g(x)}^{h(x)} p_2(y)\,dy = \int_a^b$ $\cdot\, p_1(x)\,dx\,\{P_2(h(x)) - P_2(g(x))\} = A$, A is the probability of acceptance, and hence $p(x)\,dx$ is the probability of an accepted x lying on $(x, x + dx)$.

Note. In this and other rejection methods based on it, we call the probability A of acceptance the "efficiency" of the method. Its value is irrelevant for the actual sampling rule.

R8. $p(x) = (2/\pi)^{1/2}e^{-x^2/2}$; $(0,\infty)$.

R_x. Set $x = -\ln r_1$, $y = -\ln r_2$. Accept x if $(x - 1)^2 \le 2y$.

J. Special case of R7, with $a = c = 0$, $b = d = \infty$, $p_1(x) = e^{-x}$, $p_2(y) = e^{-y}$, $P_2(y) = 1 - e^{-y}$, $g(x) = (x - 1)^2/2$, $h(x) = \infty$. (C29 is used to sample for x and y.) Specifically, we have

$$\begin{aligned}(2e/\pi)^{1/2}p_1(x)\{P_2(h(x)) - P_2(g(x))\} &= (2e/\pi)^{1/2}e^{-x}\left\{1 - \left[1 - e^{-(x-1)^2/2}\right]\right\} \\ &= (2e/\pi)^{1/2}e^{-x}e^{-(x^2-2x+1)/2} \\ &= (2/\pi)e^{-x^2/2} = p(x) \text{ as given.}\end{aligned}$$

Note. Efficiency $A = (\pi/2e)^{1/2} \approx .76$.

R9. $q(y) = e^{-y^2/2}/(2\pi)^{1/2}$; $(-\infty,\infty)$.

R_x. Sample $p(x)$ for x on $(0,\infty)$ by R8. Set $y = \pm x$ with probability 1/2.

J. Special case of C28. See also C60.

R10. $s(\nu_1) = 2e^{-\nu_1^2}/\pi^{1/2}$; $(0,\infty)$.

R_x. Sample p(x) for x on $(0,\infty)$ by R8. Set $\nu_1 = x/2^{1/2}$.

J. For the given substitution one has $s(\nu_1)\, d\nu_1 = p(x)\, dx$ as in R8. (See also C51.)

R11. $t(x) = e^{-x^2}/\pi^{1/2}$; $(-\infty,\infty)$.

R_x. Sample $s(\nu_1)$ for ν_1 on $(0,\infty)$ by R10. Set $x = \pm\, \nu_1$ with probability 1/2.

J. Special case of C28. (See also C59.)

R12. $\bar{p}(z) = C^{-1}e^{-az}\sinh(bz)^{1/2}$; $(0,\infty)$, $a,b > 0$, $C = e^{b/4a}(b\pi)^{1/2}/2a^{3/2}$ (F23).

R_x. Define $K = 1 + (b/8a)$, $L = a^{-1}\{K + (K^2 - 1)^{1/2}\}$, $M = aL - 1$. Set $x = -\ell n r_1$, $y = -\ell n r_2$. Accept x if $(y - M(x + 1))^2 \leq bLx$. For accepted x, set $z = Lx$. (After Kalos [25].)

J. The rule follows from C2, C29, and R7. In fact, for $z = Lx$, with arbitrary $L > 1/a$, and $M \equiv aL - 1 > 0$, we have

$$\begin{aligned}
\bar{p}(z)\, dz &= C^{-1}L\, dx\, e^{-aLx}(1/2)\left\{e^{(bLx)^{1/2}} - e^{-(bLx)^{1/2}}\right\} \\
&= C^{-1}(L/2)\, dx\, e^{-(M+1)x}\left\{e^{(bLx)^{1/2}} - e^{-(bLx)^{1/2}}\right\} \\
&= C^{-1}(Le^M/2)\, dx\, e^{-x}e^{-M(x+1)}\left\{e^{(bLx)^{1/2}} - e^{-(bLx)^{1/2}}\right\} \\
&= C^{-1}(Le^M/2)\, dx\, e^{-x}\left\{\left(1 - e^{-[M(x+1)+(bLx)^{1/2}]}\right)\right. \\
&\quad \left. - \left(1 - e^{-[M(x+1)-(bLx)^{1/2}]}\right)\right\} = A^{-1}\left\{e^{-x}\, dx\right\}\left\{P_2\big(h(x)\big) - P_2\big(g(x)\big)\right\}
\end{aligned}$$

as in R7, where

$$\begin{aligned}
A^{-1} &= C^{-1}(Le^M/2),\ p_1(x) = e^{-x} \text{ on } (0,\infty),\ p_2(y) = e^{-y} \text{ on } (0,\infty),\ P_2(y) \\
&= 1 - e^{-y}, \text{ and } 0 \leq g(x) \equiv M(x + 1) - (bLx)^{1/2} \\
&\leq h(x) \equiv M(x + 1) + (bLx)^{1/2} < \infty.
\end{aligned}$$

The condition for acceptance of x: $g(x) \leq y \leq h(x)$ gives the above rule. Note. The choice of values for K, L, M insure that $g(x) = M(x + 1) - (bLx)^{1/2} \geq 0$ on $(0,\infty)$. For, with $\xi = x^{1/2}$, this is equivalent to $f(\xi) \equiv \xi^2 - \frac{(bL)^{1/2}}{M}\xi + 1 \geq 0$ for $\xi > 0$. Since the parabola $f(\xi)$ opens up, with vertex at $\xi_0 = (bL)^{1/2}/2M > 0$, we see that $f(\xi) \geq 0$ iff its discriminant $(bL/M^2) - 4 \leq 0$. For $M = aL - 1$, we find $(bL/M^2) - 4 = 0$ when $aL = (K + (K^2 - 1)^{1/2})$, with $K = 1 + (b/8a)$. The choice of the (+) sign in the solution of the quadratic for aL makes $aL > 1$, as required.

R13. $p(x) = A^{-1}p_1(x)h(x)$; (a,b), $p_1(x)$ density on (a,b), $0 \leq h(x) \leq 1$.

R_x. Sample $p_1(x)$ for x on (a,b). Accept x if $r_0 \leq h(x)$.

J. Special case of R7, with $c = 0$, $d = 1$, $p_2(y) \equiv 1$ on (0,1), $P_2(y) = \int_0^y dy = y$, $g(x) \equiv 0$, $P_2(h(x)) - P_2(0) = h(x)$. Efficiency A.

<u>Note 1.</u> $1 = \int_a^b p(x)\,dx = \int_a^b A^{-1}p_1(x)h(x)\,dx \leq A^{-1}\int_a^b p_1(x)\,dx = A^{-1}$, so $A \leq 1$ is a formal consequence.

<u>Note 2.</u> The method is useful in Klein-Nishina (incoherent) and Thomson (coherent) scattering modified by form factors. Due to the nature of the latter, efficiency considerations make it expedient to use the Klein-Nishina density for $p_1(x)$ and the form factor for h(x) in incoherent scattering, whereas in coherent scattering, $p_1(x)$ is based on the form factor, and h(x) on the Thomson cross section. For further details, see [5; Part II].

<u>Note 3.</u> If a given density p(x) on (a,b) is of form $p(x) = s(x)t(x)$, where $s(x), t(x) \geq 0$, $S = \int_a^b s(x)\,dx$ is positive and finite, and t(x) is bounded on (a,b) with $0 \leq t(x) \leq \hat{t}$, one can always write $p(x) = (S\hat{t})(s(x)/S)(t(x)/\hat{t})$ in the form of R13. (Efficiency $1/S\hat{t}$).

<u>Note 4.</u> If p(x) is a given density on (a,b), and one wishes to sample it by R13, using a <u>particular</u> density $p_1(x)$, one can always write $p(x) = (M)(p_1(x))\left(\frac{p(x)/p_1(x)}{M}\right)$ in the form of R13, provided $M = \max_{(a,b)} p(x)/p_1(x)$ is finite. See R14, 15, 16, 17 for $p_1(x)$ uniform.

Note 5. Analysis of the assignments to $(x, x + dx)$ according to the required number of trials shows that the total probability of such an assignment is $p_1(x)\,dx\,h(x)\{1 + (1 - A) + (1 - A)^2 + \ldots\}$ $= p_1(x)\,dx\,h(x)(1/A) = p(x)\,dx$, where A is the chance of assignment to some interval, and $1 - A$ the chance of assignment to no interval. The total chance of assignment on the ν-th trial is $(1 - A)^{\nu-1}A$, with sum $A + (1 - A)A + (1 - A)^2A + \ldots = A(1/A) = 1$. The expected number of trials for assignment is $\sum_1^\infty \nu(1 - A)^{\nu-1}A = 1/A$, the inverse efficiency.

Note here that, with $x = 1 - A$, $1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \ldots) = \frac{d}{dx}(1 - x)^{-1} = (1 - x)^{-2}$.

R14. $p(x) = A^{-1}(b - a)^{-1}(p(x)/\hat{p})$; (a,b), $\hat{p} = \max_{(a,b)} p(x)$, $A = 1/(b - a)\hat{p}$ efficiency.

R_x. Accept $x = a + r_0(b - a)$ if $r_1 \leq p(x)/\hat{p}$.

J. Special case of R13, with stipulated $p_1(x) = (b - a)^{-1}$ uniform, and $h(x) = p(x)/\hat{p} \leq 1$.

R15. $p(x) = F^{-1}f(x)$; $F = \int_a^b f(x)\,dx$.

R_x. Accept $x = a + r_0(b - a)$ if $r_1 \leq f(x)/M$, where $M = \max_{(a,b)} f(x)$.

J. One can write $p(x) = A^{-1}(b - a)^{-1}(f(x)/M)$, where $A = \int_a^b (b - a)^{-1}f(x) \cdot dx/M = F/(b - a)M$, $(b - a)^{-1}$ is the stipulated uniform density, and $f(x)/M \leq 1$. The rule follows from R13.

Note 1. A lower bound on the efficiency A may be obtained as follows. Let $m = \min_{(a,b)} f(x)$, $M = \max_{(a,b)} f(x)$ (as above). Then

$$A = (b - a)^{-1} \int_a^b (f(x)/M)\,dx \geq (b - a)^{-1} \int_a^b [(f(x) - m)/(M - m)]\,dx.$$

And of course, $A \geq (b - a)^{-1}(m/M) = m/(b - a)M$.

Note 2. The value of F is irrelevant both in the rule and the lower bound estimate of A.

R16. $p(\phi) = F^{-1}f(\phi)$, $f(\phi) = K - B\cos^2\phi$; $(0,2\pi)$, $K > B > 0$.

R_x1. Accept $\phi = 2\pi r_0$ if $r_1 \leq f(\phi)/K$.

J1. Special case of R15.

Note 1. Since $m = \min f(\phi) = K - B$, and $M = \max f(\phi) = K$, one may use R15, Note, to show that efficiency

$$A \gtrsim (1/2\pi)\int_0^{2\pi} (1 - \cos^2\phi)\, d\phi = 1/2.$$

R_x2. Obtain $\cos\phi$ from R3, and accept if next $r \leq f(\phi)/K$.

J2. This is just a way of avoiding computation of $\cos\phi$.

Note 2. The density $p(\phi)$ occurs in polarized Compton scattering [10,15], where ϕ itself is not required.

R17. $p(\phi) = F^{-1}f(\phi)$; $f(\phi) = K - S^2(Q\cos 2\phi + U\sin 2\phi)$; $(0,2\pi)$, $K > H \equiv S^2(Q^2 + U^2)^{1/2}$.

R_x1. Accept $\phi = 2\pi r_0$ if $r_1 \leq f(\phi)/(K + H)$.

J1. We write $f(\phi) = K - S^2(Q^2 + U^2)^{1/2}\{[Q/(Q^2 + U^2)^{1/2}]\cos 2\phi + [U/(Q^2 + U^2)^{1/2}]\sin 2\phi\} = K - H(\cos 2\phi_0 \cos 2\phi + \sin 2\phi_0 \sin 2\phi) = K - H\cos\{2(\phi - \phi_0)\}$, where $2\phi_0$ is uniquely defined on $(0,2\pi)$ by the relations $\cos 2\phi_0 = Q/(Q^2 + U^2)^{1/2}$, $\sin 2\phi_0 = U/(Q^2 + U^2)^{1/2}$. Hence $m = \min f(\phi) = K - H$, $M = \max f(\phi) = K + H$, so $p(\phi) = A^{-1}(1/2\pi)(f(\phi)/M)$, with $A = F/2\pi M$. The rule follows from the latter and R15, while we find from the Note in R15 that the efficiency

$$A \geq (1/2\pi)\int_0^{2\pi} [(f(\phi) - m)/(M - m)]\, d\phi = (1/4\pi)\int_0^{2\pi} (1 - \cos(2\phi - 2\phi_0))\, d\phi = (1/4\pi)(2\pi - 0) = 1/2.$$

R_x2. Obtain $\cos\phi$, $\sin\phi$ from R3, compute $\cos 2\phi = \cos^2\phi - \sin^2\phi$, $\sin 2\phi = 2\sin\phi\cos\phi$. Accept ϕ if next $r \leq f(\phi)/(K + H)$.

J2. This is only a way of avoiding computation of cos 2ϕ, sin 2ϕ from angle $2\phi = 4\pi r_0$.

Note. The density $p(\phi)$ occurs in polarized Compton scattering [10,15], where ϕ itself is not required.

R18. $t(x) = \begin{cases} h(x - a)/(b - a); & (a,b) \\ h(c - x)/(c - b); & (b,c), \end{cases}$ $a < b < c$, $h \equiv 2/(c - a)$.

R_x. One follows the steps:

1. Generate next two random numbers r, r'.
2. Set $x = a + (c - a)r$.
3. Accept x if $x \leq b$ and $r' \leq (x - a)/(b - a)$, or if $x > b$ and $r' \leq (c - x)/(c - b)$. Otherwise return to (1).

J. The rule may be regarded as a special case of R13, if we write $t(x) = 2(c - a)^{-1}(t(x)/h)$, the efficiency being obviously 1/2. See also C114.

R19. $p(x) = (2/\pi) \sin^2 x/x^2$; $(0,\infty)$. (See F22.)

R_x. Define $A_1 = \int_0^1 p(x)\, dx$ ($\approx .57$), $A_2 = 1 - A_1$. One follows the steps:

1. If $r \leq A_1$ go to (2). Otherwise go to (3).
2. Sample density $(2/\pi A_1)(1/1)(\sin^2 x/x^2) \equiv E_1^{-1} p_1(x) h_1(x)$ for x on $(0,1)$ by R13, i.e., set $x = r'$, and accept x if $r'' \leq \sin^2 x/x^2$. Otherwise iterate (2).
3. Sample density $(2/\pi A_2)(1/x^2)(\sin^2 x) \equiv E_2^{-1} p_2(x) h_2(x)$ for x on $(1,\infty)$ by R13, i.e., set $x = 1/r'$, and accept x if $r'' \leq \sin^2 x$. Otherwise iterate (3).

J. The rule results from C113, since we may regard $p(x)$ as the composite function

$$p(x) = \begin{cases} (2/\pi)(1/1)(\sin^2 x/x^2); & (0,1) \\ (2/\pi)(1/x^2)(\sin^2 x); & (1,\infty), \end{cases}$$

as in C113. Note that $h_1(x) = \sin^2 x/x^2 \leq 1$ on $(0,1)$, and $h_2(x) = \sin^2 x \leq 1$ on $(1,\infty)$. The settings of x result from C1, with $r' = \int_0^x dx$, $x = r'$, and $r' = \int_x^{\infty} dx/x^2$, $x = 1/r'$.

R20. $p(x) = \sum_1^J \alpha_j p_j(x) h_j(x)$; (a,b), $\alpha_j > 0$, $p_j(x)$ density on (a,b), $0 \leq h_j(x) \leq 1$ for x on (a,b).

R_x1. (J finite or infinite.) Define $A_j = \int_a^b \alpha_j p_j(x) h_j(x)\, dx$.

Set $K = \min\left\{k;\ \sum_1^k A_j \geq r_0\right\}$. Then:

1. Sample density $p_K(x)$ for x on (a,b).
2. Accept x if next $r \leq h_K(x)$. Otherwise return to (1).

J1. The rule is an obvious consequence of C3 and R13, since we may write

$$p(x) = \sum_1^J a_j(x) = \sum_1^J A_j(a_j(x)/A_j) = \sum_1^J A_j\{(A_j^{-1}\alpha_j)(p_j(x))(h_j(x))\}.$$

Note 1. The probability of sampling the j-th density $a_j(x)/A_j$ is A_j, while the efficiency of sampling this density is A_j/α_j. Hence $\sum_1^J A_j(A_j/\alpha_j)$ is the average efficiency of the rule. However, α_j/A_j is the expected number of trials for acceptance in sampling density a_j/A_j, so $\sum_1^J A_j(\alpha_j/A_j) = \sum_1^J \alpha_j$ is the average of the expected number of trials (finite or infinite. See Note 3.).

R_x2. (J finite.) Define $\sigma = \sum_1^J \alpha_j$. Then:

1. Generate next two random numbers r, r'.

2. Set $K = \min\left\{k; \sum_{1}^{k} \alpha_j \geq r\sigma\right\}$.

3. Sample density $p_K(x)$ for x on (a,b).

4. Accept x if $r' \leq h_K(x)$. Otherwise return to (1).

J2. The total probability of accepting x on $(x, x + dx)$ is $\sum_{1}^{J} (\alpha_j/\sigma) p_j(x)\, dx \cdot h_j(x)$, the total chance of acceptance for all x being the integral $1/\sigma$. Hence the relative probability of an accepted x lying on $(x, x + dx)$ is $\left\{\sum_{1}^{J} (\alpha_j/\sigma) p_j(x)\, dx \cdot h_j(x)\right\} \Big/ (1/\sigma) = p(x)\, dx$, as required. The overall efficiency of the rule is $1/\sigma$, σ being the expected number of trials for acceptance.

Note 2. If $p_j(x)$, $j = 1,2, \ldots$ are arbitrary densities on (a,b), then $p(x) = \sum_{1}^{\infty} (1/j) \cdot p_j(x) \cdot (6/\pi^2 j)$ is a density on (a,b) of the form in R20, for which $\sum_{1}^{\infty} \alpha_j = \sum_{1}^{\infty} (1/j)$ does not converge.

Note 3. As in Note 3 of R13, any density of the form $p(x) = \sum s_j(x) t_j(x)$ may be written in the form $p(x) = \sum (S_j \hat{t}_j) \cdot (s_j(x)/S_j)(t_j(x)/\hat{t}_j)$ of R20, subject to the obvious conditions.

R21. $p(\nu) = C_1^{-1} \nu^{n-1}/(\Lambda^{-1} e^{\nu} + 1)$; $(0,\infty)$, $0 < \Lambda \leq 1$, $n \in \{3/2, 2, 5/2, 3, \ldots\}$,

$C_1 = \zeta_a(\Lambda,n)\Gamma(n)$, where $\zeta_a(\Lambda,n) = \sum_{1}^{\infty} (-1)^{j+1} \Lambda^j / j^n$. (See F11.)

Rx. Sample $q(\nu)$ for ν on $(0,\infty)$ by C74. Accept ν if $r_1 \geq \Lambda e^{-\nu}$.

J. The rule follows from R13, since we may write $p(\nu) \equiv C_1^{-1} \nu^{n-1} \Lambda e^{-\nu}/(1 + \Lambda e^{-\nu}) = (C_1^{-1} C)\{C^{-1} \nu^{n-1} \Lambda e^{-\nu}/(1 - \Lambda^2 e^{-2\nu})\}(1 - \Lambda e^{-\nu}) = A^{-1}\{q(\nu)\} h(\nu)$, where $A = C_1C^{-1} = \zeta_a(\Lambda,n)/\zeta_u(\Lambda,n)$, $q(\nu) = C^{-1}\nu^{n-1}\Lambda e^{-\nu}/(1 - \Lambda^2e^{-2\nu})$ is the density in C74, and $h(\nu) = 1 - \Lambda e^{-\nu} \leq 1$. Using R13, we accept ν from C74 in case $r_0 \leq 1 - \Lambda e^{-\nu}$, equivalently $\Lambda e^{-\nu} \leq 1 - r_0 = r_1$.

R22. $p(y) = C^{-1}y^{1/2}/(e^{y-\eta} + 1)$; $(0,\infty)$, $-\infty < \eta \leq 50$.

R_x. Case I. $-\infty < \eta \leq 5/2$. Sample density $y^{1/2}e^{-y}/\Gamma(3/2)$ for y on $(0,\infty)$ by C64. Accept y if $r \leq 1/(1 + e^{\eta-y})$.

JI. We write

$$p(y) = (C^{-1}\Gamma(3/2)e^{\eta})(y^{1/2}e^{-y}/\Gamma(3/2))(1 + e^{\eta-y})^{-1},$$

which is of the form in R13, with $A = C/\Gamma(3/2)e^{\eta}$, the efficiency, $p_1(y) = y^{1/2}e^{-y}/\Gamma(3/2)$, a density on $(0,\infty)$, and $h(y) = 1/(1 + e^{\eta-y}) < 1$, the acceptance factor. The efficiency in Case I is never less than 30%, and drops below 50% only for a short range of η values around $\eta = 2$.

R_x. Case II. $5/2 < \eta \leq 50$. Define $A_1 = \int_0^{\eta} p(y)\, dy$, $A_2 = 1 - A_1$. If $r_1 \leq A_1$, sample density $(3/2)y^{1/2}/\eta^{3/2}$ for $y = \eta r_0^{2/3}$ on $(0,\eta)$; accept y with probability $(e^{-\eta} + 1)/(e^{y-\eta} + 1)$. If $r_1 > A_1$, sample the residual Γ- density $ye^{-y}/(\eta + 1)e^{-\eta}$ for y on (η,∞) by C108; accept y with probability $h/y^{1/2}(1 + e^{\eta-y})$, where $h = \min_{\eta<y<\infty} y^{1/2}(1 + e^{\eta-y})$.

JII. We regard $p(y)$ as a composite function

$$p(y) = \begin{cases} a_1(y); & (0,\eta) \\ a_2(y); & (\eta,\infty) \end{cases}$$

as in C113, with $A_1 = \int_0^{\eta} p(y)\, dy$. We may then sample $a_1(y)/A_1$ for y on $(0,\eta)$ with probability A_1, and $a_2(y)/A_2$ for y on (η,∞) with probability $A_2 = 1 - A_1$. Both densities $a_i(y)/A_i$ may be written in the form of R13 and sampled accordingly. Specifically, we write

$$a_1(y)/A_1 = \{A_1^{-1}C^{-1}(2/3)\eta^{3/2}(e^{-\eta} + 1)^{-1}\}\{(3/2)y^{1/2}/\eta^{3/2}\}\{(e^{-\eta} + 1)/(e^{y-\eta} + 1)\}, \text{ and } a_2(y)/A_2 = \{A_2^{-1}C^{-1}(\eta + 1)h^{-1}\}\{ye^{-y}/\Gamma_{\eta}\}\{h/y^{1/2}(1 + e^{\eta-y})\},$$

where $\Gamma_\eta = (\eta + 1)e^{-\eta}$, and $h = \min_{\eta<y<\infty} y^{1/2}(1 + e^{\eta-y})$. In Case II, the efficiency on $(0,\eta)$ always exceeds 1/2, while on (η,∞) it drops very slowly from 89% at $\eta = 3$ to 71% at $\eta = 50$. How far the method can be extended above $\eta = 50$ we do not know.

Note 1. In practice, C is a given physical constant determined by the electron density and temperature, and the degeneracy parameter $\eta = \eta(C)$ is determined so that

$$I(\eta) = \int_0^\infty y^{1/2}\, dy/(e^{y-\eta} + 1) = C.$$

The function $I(\eta)$ is well tabulated (see references in [6]), so that, in a given physical case, the values of C and η are known.

Note 2. Details of the method, including tables of norming constants C, A_1, efficiencies, and minima h, as functions of η, are given in [6].

R23. $\overline{p}(x) = E_\xi^{-1}x^{n-1}e^{-\xi(x^2+1)^{1/2}}$; $(0,\infty)$, $\xi > 0$, $n = 2,3,4, \ldots$, $E_\xi = (\Gamma(n/2)/\Gamma(1/2))(2/\xi)^{(n-1)/2}K_{\frac{n+1}{2}}(\xi)$. (See F14.)

Rx. Sample $q(y) = D_\xi^{-1}y^{n-1}e^{-\xi y}$ for y on $(1,\infty)$ by C106. Accept y with probability $(1 - (1/y^2))^{(n/2)-1}$, $n \geq 2$. For accepted y, set $x = (y^2 - 1)^{1/2}$.

J. Under the transformation $x = (y^2 - 1)^{1/2}$, one has $\overline{p}(x)\, dx = E_\xi^{-1}y(y^2 - 1)^{(n/2)-1}e^{-\xi y}\, dy = (D_\xi/E_\xi)(y^{n-1}e^{-\xi y}\, dy/D_\xi)(1 - (1/y^2))^{(n/2)-1}$. Hence by C2, we can sample the latter for y on $(1,\infty)$, and set $x = (y^2 - 1)^{1/2}$. But this is of the form $(A^{-1})(p_1(y)\, dy)(h(y))$ as in R13, and the rule follows from C106.

Note 1. The efficiency is $A = E_\xi/D_\xi$. For $n = 2$, acceptance is certain, and $D_\xi = E_\xi$ is easily verified using F3C and F13C.

Note 2. For $n = 3$, $\overline{p}(x)$ is the Maxwell-Juttner relativistic momentum density [8].

R24. $\bar{p}(x) = A^{-1}x^{n-1}/(\Lambda^{-1}e^{a(x^2+1)^{1/2}} + 1)$; $(0,\infty)$, $a > 0$, $0 < \Lambda \leq 1$, $n = 2,3,4, \ldots$, $A = 2^{(n-1)/2} \frac{\Gamma(n/2)}{\Gamma(1/2)} \sum_{1}^{\infty} (-1)^{j+1}\Lambda^j K_{\frac{n+1}{2}}(ja)/(ja)^{(n-1)/2}$.

(See F15.)

R_x. Sample $\bar{q}(\nu)$ for ν on $(1,\infty)$ by C109. Accept ν with probability $(1 - \Lambda e^{-a\nu})(1 - (1/\nu^2))^{(n/2)-1}$. For accepted ν, set $x = (\nu^2 - 1)^{1/2}$.

J. Under the preceding transformation, one has

$$\bar{p}(x)\,dx = A^{-1}\nu(\nu^2 - 1)^{(n/2)-1}\,d\nu/(\Lambda^{-1}e^{a\nu} + 1)$$

$$= A^{-1} \cdot \frac{\nu^{n-1}\,d\nu}{\Lambda^{-1}e^{a\nu} + 1}(1 - (1/\nu^2))^{(n/2)-1}$$

$$= \frac{A^{-1}\nu^{n-1}\Lambda e^{-a\nu}\,d\nu}{1 + \Lambda e^{-a\nu}}(1 - (1/\nu^2))^{(n/2)-1}$$

$$= \left(\frac{D}{A}\right)\left\{\frac{\nu^{n-1}\Lambda e^{-a\nu}\,d\nu}{D(1 - \Lambda^2 e^{-2a\nu})}\right\}[(1 - \Lambda e^{-a\nu})(1 - (1/\nu^2))^{(n/2)-1}],$$

$n \geq 2$, which is of the standard form in R13, the density in braces being the $\bar{q}(\nu)$ of C109. The rule follows from C2, R13, and C109.

R25. $q(x) = (\phi/\pi)^{1/2}x^{-3/2}e^{-\phi(x-1)^2/x}$; $(0,\infty)$, $\phi > 0$.

R_x. Sample $e^{-z^2}/\pi^{1/2}$ for z on $(-\infty,\infty)$ by C59 or R11. Accept z if $r \leq (1/2) \cdot (1 - z/(z^2 + 4\phi)^{1/2})$. (See Note 1.) For accepted z, set $x = [(z^2 + 4\phi)^{1/2} + z]/[(z^2 + 4\phi)^{1/2} - z]$.

J. The function $z = (\phi/x)^{1/2}(x - 1)$ increases from $-\infty$ to ∞ as x increases from 0 to ∞. Moreover, $dz = \phi^{1/2}(x + 1)\,dx/2x^{3/2}$. Hence $q(x)\,dx = 2(e^{-z^2}/\pi^{1/2})(x + 1)^{-1}\,dz$. To evaluate $(x + 1)^{-1}$ in terms of z, we proceed indirectly thus: From $z^2 = \phi(x - 1)^2/x$ follows $z^2 + 4\phi = \phi[(x - 1)^2/x + 4] = \phi(x + 1)^2/x$. Hence

$$z = (\phi/x)^{1/2}(x - 1), \text{ and } (z^2 + 4\phi)^{1/2} = (\phi/x)^{1/2}(x + 1).$$

By division, we have $(x - 1)/(x + 1) = z/(z^2 + 4\phi)^{1/2}$, whence $1/(x + 1) = (1/2)[1 - (x - 1)/(x + 1)] = (1/2)[1 - z/(z^2 + 4\phi)^{1/2}]$. Therefore, we may write $q(x)\,dx = 2(e^{-z^2}/\pi^{1/2})(1/2)(1 - z/(z^2 + 4\phi)^{1/2})\,dz \equiv p(z)\,dz$ on $(-\infty,\infty)$. By C2, we may sample the latter $p(z)$ for z on $(-\infty,\infty)$ and set $x = x(z)$. Solution of the preceding equation

$$1/(x + 1) = (1/2)[1 - z/(z^2 + 4\phi)^{1/2}]$$

for x gives the formula $x = x(z)$ of the rule. But $p(z)$ is of the form $p(z) = A^{-1}p_1(z)h(z)$ in R13, where $A \equiv 1/2$, the efficiency, $p_1(z) = e^{-z^2}/\pi^{1/2}$ is a density on $(-\infty,\infty)$, and $h(z) = (1/2)(1 - z/(z^2 + 4\phi)^{1/2})$ satisfies $0 < h(z) < 1$.

Hence, by R13, we sample $p_1(z)$ for z on $(-\infty,\infty)$, and accept z with probability $h(z)$, as in the rule.

Note 1. The acceptance condition $r \leq (1/2)(1 - z/(z^2 + 4\phi)^{1/2})$ may be interpreted thus:

1. If $z \geq 0$, accept z iff $r \leq 1/2$ and $(1 - 2r)^2 \geq z^2/(z^2 + 4\phi)$.
2. If $z < 0$, accept z iff $(1 - 2r)^2 \leq z^2/(z^2 + 4\phi)$.

Note 2. For testing purposes, we have included below an evaluation of the Wald distribution function

$Q(x) = \int_0^x q(\xi)\,d\xi$ in terms of the well tabulated normal distribution,

$\Phi(y) = (1/2\pi)^{1/2} \int_{-\infty}^{y} e^{-\eta^2/2}\,d\eta$. For convenience we work with

$$G(z) = (1/\pi)^{1/2} \int_{-\infty}^{z} e^{-\zeta^2}\,d\zeta = \Phi(2^{1/2}z).$$

From (J) above, with $z = (\phi/x)^{1/2}(x - 1)$, we obtain

$$Q(x) = \int_0^x q(\xi)\,d\xi = (1/\pi)^{1/2} \int_{-\infty}^{z} (1 - \zeta/(\zeta^2 + 4\phi)^{1/2})e^{-\zeta^2}\,d\zeta = G(z)$$

$- H(z)$, where $H(z) = (1/\pi)^{1/2} \int_{-\infty}^{z} [\zeta/(\zeta^2 + 4\phi)^{1/2}]e^{-\zeta^2}\,d\zeta$. For a fixed z

≤ 0, the substitution $\eta = -(\zeta^2 + 4\phi)^{1/2}$ in $H(z)$ gives $H(z) = -e^{4\phi}G(-(z^2 + 4\phi)^{1/2})$. Since the integrand of $H(z)$ is an odd function, we know $H(-z) = H(z)$, so $H(z) = -e^{4\phi}G(-(z^2 + 4\phi)^{1/2})$ for <u>all</u> z on $(-\infty,\infty)$. Hence for all x on $(0,\infty)$, we have $Q(x) = G(z) + e^{4\phi}G(-(z^2 + 4\phi)^{1/2})$. From (J) we recall that $z = (\phi/x)^{1/2}(x - 1)$, and $(z^2 + 4\phi)^{1/2} = (\phi/x)^{1/2}(x + 1)$, so that $Q(x) = G((\phi/x)^{1/2}(x - 1)) + e^{4\phi}G(-(\phi/x)^{1/2}(x + 1)) = \Phi((2\phi/x)^{1/2} \cdot (x - 1)) + e^{4\phi}\Phi(-(2\phi/x)^{1/2}(x + 1))$. See [22, v.2; p. 141.]

<u>R26. $q(R) = (1 - R^2)^{(T-1)/2}/(1 + \rho^2 - 2\rho R)^{T/2}B(1/2,(T + 1)/2)$;</u> $(-1,1)$, $0 < \rho < 1$, $T \in \{1,2,3, \ldots\}$.

R_x1. Sample $B(v) = v^{n-1}(1 - v)^{n-1}/B(n,n)$ for v on $(0,1)$ with $n = (T + 1)/2$ by C75. Accept v with probability $h(v) = (1 - \rho)^T/((1 + \rho)^2 - 4\rho v)^{T/2}$. For accepted v, set $R = 2v - 1$.

J1. For $R = 2v - 1$, one has

$$q(R)\ dR = \frac{2^T B((T + 1)/2,\ (T + 1)/2)}{B(1/2,(T + 1)/2)(1 - \rho)^T} \cdot \frac{v^{(T-1)/2}(1 - v)^{(T - 1)/2}dv}{B((T + 1)/2,(T + 1)/2)}$$

$$\cdot \frac{(1 - \rho)^T}{((1 + \rho)^2 - 4\rho v)^{T/2}} \equiv (1 - \rho)^{-T}B(v)\ dv \cdot h(v),$$

and the rule follows from R13 and C2.

<u>Note 1.</u> $$\frac{2^T B((T + 1)/2,(T + 1)/2)}{B(1/2,(T + 1)/2)} = \frac{2^T\Gamma^2((T + 1)/2)}{\Gamma(T + 1)} \cdot \frac{\Gamma((T + 2)/2)}{\Gamma(1/2)\Gamma((T + 1)/2)}$$

$= 1$ by F4E with $m = (T + 1)/2$.

<u>Note 2.</u> $$\min_{v\in(0,1)} (1 + \rho)^2 - 4\rho v = (1 + \rho)^2 - 4\rho = (1 - \rho)^2.$$

R_x2. Sample $b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$ for z on $(0,\infty)$, by C75, with $m = (T + 1)/2$, $n = 1/2$. Accept z with probability $(1 + \rho)^T/[(1 + \rho)^2 + (1 - \rho)^2 z]^{T/2} \equiv h(z)$. For accepted z, set $R = (z - 1)/(z + 1)$.

J2. For the latter substitution one finds

$$q(R)\ dR = \frac{2^T}{(1 + \rho)^T} \cdot \frac{z^{(T-1)/2}\ dz}{(1 + z)^{(T+2)/2}B((T + 1)/2,1/2)}$$

$$\cdot \frac{(1 + \rho)^T}{[(1 + \rho)^2 + (1 - \rho)^2 z]^{T/2}} \equiv \left(\frac{1 + \rho}{2}\right)^{-T} \cdot b(z)\ dz \cdot h(z),$$

and the rule follows from R13 and C2.

Note 3. $\min_{z\varepsilon(0,\infty)} (1 + \rho)^2 + (1 - \rho)^2 z = (1 + \rho)^2$.

Note 4. The rules are only practical for small T, their respective efficiencies being $(1 - \rho)^T$ and $\left(\frac{1 + \rho}{2}\right)^T$. For $p \leq 1/3$ use R_x1, for $\rho > 1/3$, use R_x2. The efficiencies are then both minimal for $\rho = 1/3$, where $(2/3)^T$ is the common efficiency.

R27. $p(x) = x^{m-1}e^{-x}/\Gamma(m)$; $(0,\infty)$, $m > 0$, $m \notin \{1/2,1,3/2,2, \ldots\}$.

R_x1. Let $m = H + R$, where $H \varepsilon \{0,1/2,1,3/2, \ldots\}$, and $0 < R < 1/2$. Define $n = 1/2 - R$, $0 < n < 1/2$. Set $s = r_1^{1/m}$, $t = r_2^{1/n}$, and iterate until $s + t \leq 1$. For accepted s,t, set $v = s/(s + t)$. Sample $u^{H-1/2}e^{-u}/\Gamma(H + 1/2)$ for u on $(0,\infty)$ by C45 or C64. Set $x = uv$. (Jöhnk.)

J1. The rule results from the following remarks:

A. Under the transformation $x = uv$, $y = u(1 - v)$, with Jacobian $-u$, and inverse $u = x + y$, $v = x/(x + y)$, one finds that

$$\frac{x^{m-1}e^{-x}\,dx}{\Gamma(m)} \cdot \frac{y^{n-1}e^{-y}\,dy}{\Gamma(n)} = \frac{u^{m+n-1}e^{-u}\,du}{\Gamma(m + n)} \cdot \frac{v^{m-1}(1 - v)^{n-1}\,dv}{B(m,n)}$$

on $(0,\infty) \times (0,1)$. (See Fig. 1.) Hence by C2, we may sample the latter two densities and set $x = uv$. The first is possible by C45 or C64, since $m + n = H + 1/2 \varepsilon \{1/2,1,3/2,2, \ldots\}$. It remains to sample the second for v on (0,1).

B. For the density $f(s,t) = ms^{m-1}nt^{n-1}$ on $(0,1) \times (0,1)$, we find for s,t the probability of acceptance

$$E = P\{s + t \leq 1\} = \int_0^1 ms^{m-1}\,ds \int_0^{1-s} nt^{n-1}\,dt = m\int_0^1 s^{m-1}(1 - s)^n\,ds$$

$$= mB(m,n + 1) = m\Gamma(m)\Gamma(n + 1)/\Gamma(m + n + 1) = \Gamma(m + 1)\Gamma(n + 1)/\Gamma(m + n + 1).$$

Hence the accepted pairs (s,t) have the conditional density function $g(s,t) = E^{-1}f(s,t) = E^{-1}ms^{m-1}nt^{n-1}$ for $s,t > 0$, $s + t \leq 1$. Under g(s,t), the density for the value of the function $v \equiv s/(s + t)$ is found to be $B(v) = v^{m-1}(1 - v)^{n-1}/B(m,n)$ on (0,1). (See Note 1.) Hence we may sample g(s,t) for s,t by rejection technique, and set $v = s/(s + t)$ for accepted s,t, as in the rule.

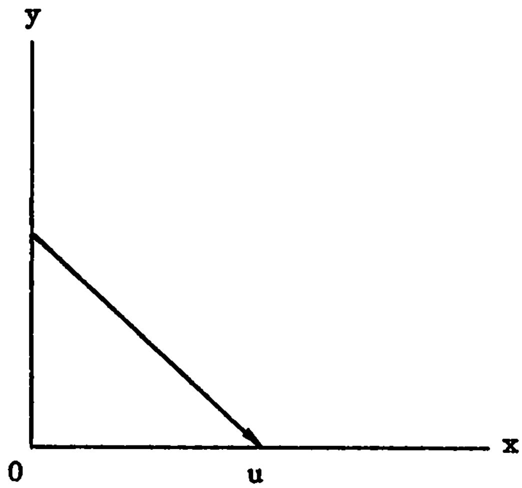
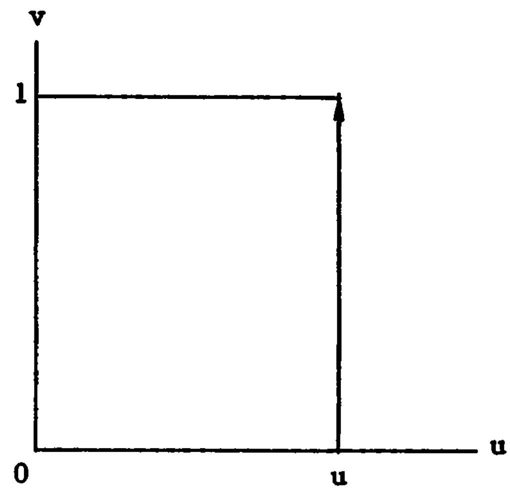


Fig. 1.



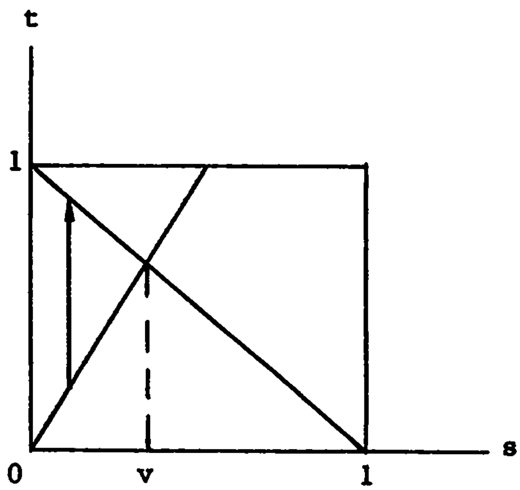


Fig. 2.

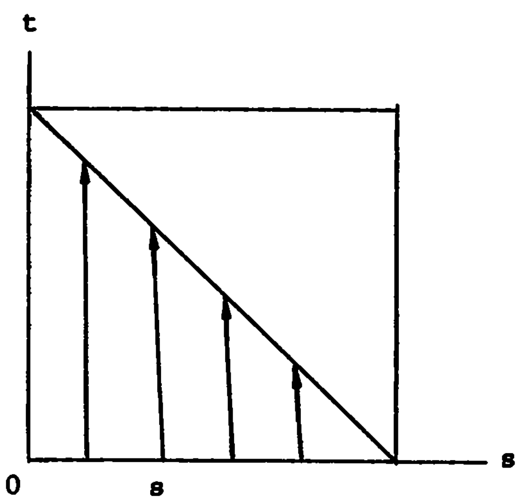


Fig. 3.

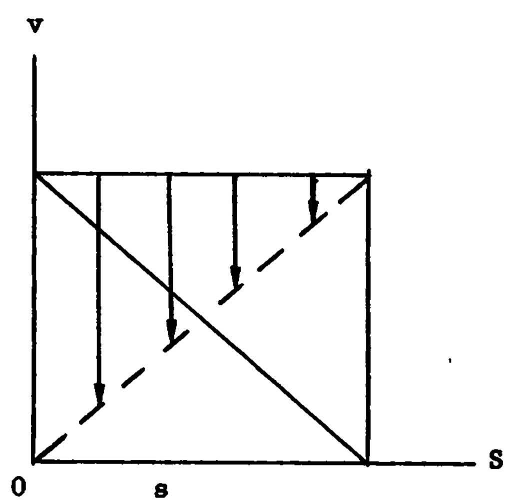


Fig. R27.

Note 1. The density B(v) for $v = s/(s + t)$ may be verified by either of the following two methods.

(i). The points (s,t) of the unit square for which $s + t \leq 1$ and $s/(s + t) \leq v$ are those below the line $t = 1 - s$, and above the line $t = s(1 - v)/v$, which intersect at $s = v$. (Fig. 2.) Note that $s/(s + t) \leq v$ is equivalent to $(s + t)/s \geq 1/v$, or $t \geq s(1 - v)/v$. Hence

$$\frac{d}{dv} \int_{\{s/(s+t)\leq v\}} g(s,t)\, ds\, dt = E^{-1} \frac{d}{dv} \int_0^v ms^{m-1}\, ds \int_{s(1-v)/v}^{1-s} nt^{n-1}\, dt$$

$$= E^{-1} \frac{d}{dv} \int_0^v ms^{m-1}\, ds\{(1 - s)^n - s^n(1 - v)^n/v^n\}$$

$$= E^{-1}m\left\{v^{m-1}(1 - v)^n - \frac{d}{dv} v^{-n}(1 - v)^n v^{m+n}/(m + n)\right\}$$

$$= E^{-1}m\left\{v^{m-1}(1 - v)^n - \frac{d}{dv} v^m(1 - v)^n/(m + n)\right\}$$

$$= E^{-1}m\left\{v^{m-1}(1 - v)^n - \left[\frac{mv^{m-1}(1 - v)^n - nv^m(1 - v)^{n-1}}{m + n}\right]\right\}$$

$$= E^{-1}mv^{m-1}(1 - v)^{n-1}\left\{1 - v - \frac{m}{m + n}(1 - v) + \frac{nv}{m + n}\right\}$$

$$= E^{-1}mv^{m-1}(1 - v)^{n-1}\left\{1 - \frac{m}{m + n}\right\}$$

$$= \frac{\Gamma(m + n + 1)}{\Gamma(m + 1)\Gamma(n + 1)} \cdot \frac{mn}{m + n} v^{m-1}(1 - v)^{n-1}$$

$$= \frac{\Gamma(m + n)}{\Gamma(m)\Gamma(n)} v^{m-1}(1 - v)^{n-1} = v^{m-1}(1 - v)^{n-1}/B(m,n) \quad .$$

The method of F1 may also be used.

(ii). Under the transformation $s = S$, $t = S(1 - v)/v$, with Jacobian $-S/v^2$, and inverse $S = s$, $v = s/(s + t)$, transforming the (s,t) region with $s > 0$, $t > 0$, $s + t \leq 1$ into the (S,v) region with $0 < S < 1$, $S \leq v < 1$ (Fig. 3), one finds

$$g(s,t)\, ds\, dt = E^{-1}mnS^{m-1}S^{n-1}(1 - v)^{n-1}v^{-(n-1)}Sv^{-2}\, dS\, dv = E^{-1}mnS^{m+n-1} \cdot v^{-(n+1)}(1 - v)^{n-1}\, dS\, dv \equiv h(S,v)\, dS\, dv,$$ with marginal v-density

$$\int_0^v h(S,v)\, dS = E^{-1}mn(m + n)^{-1}v^{m-1}(1 - v)^{n-1} = v^{m-1}(1 - v)^{n-1}/B(m,n)$$

$= B(v)$. Hence $B(v)$ is the density of $v = s/(s + t)$ under the density $g(s,t)$.

R_x2. Define H,R,n as in R_x1. If $H \geq 1/2$, sample $\xi^{R-1}e^{-\xi}/\Gamma(R)$ for ξ on $(0,\infty)$ as in R_x1 (i.e., with $H = 0$, $m = R$.) Sample $(\xi')^{H-1}e^{-\xi'}/\Gamma(H)$ for ξ' on $(0,\infty)$ by C45 or C64. Set $x = \xi + \xi'$. (Jöhnk.)

J2. The density of the function $x = \xi + \xi'$ under the density

$$p_1(\xi)p_2(\xi') = \frac{\xi^{R-1}e^{-\xi}}{\Gamma(R)} \cdot \frac{(\xi')^{H-1}e^{-\xi'}}{\Gamma(H)} \text{ is, by C9,}$$

$$\int_0^x p_1(\xi)p_2(x - \xi)\, d\xi = \int_0^x \frac{\xi^{R-1}e^{-\xi}}{\Gamma(R)} \cdot \frac{(x - \xi)^{H-1}e^{(x-\xi)}}{\Gamma(H)}\, d\xi$$

$$= \frac{e^{-x}}{\Gamma(R)\Gamma(H)} \int_0^x \xi^{R-1}(x - \xi)^{H-1}d\xi = \frac{e^{-x}x^{R+H-1}}{\Gamma(R)\Gamma(H)} \int_0^1 \eta^{R-1}(1 - \eta)^{H-1}d\eta$$

$$= \frac{x^{R+H-1}e^{-x}}{\Gamma(R)\Gamma(H)} \cdot B(R,H) = \frac{x^{R+H-1}e^{-x}}{\Gamma(R + H)} = \frac{x^{m-1}e^{-x}}{\Gamma(m)} = p(x),$$ as in R27. Hence the rule follows from C9.

<u>Note 2</u>. The probability $E = \Gamma(m + 1)\Gamma(n + 1)/\Gamma(m + n + 1)$ of acceptance of s,t in R_x1 becomes small for large m, but is high for $0 < m = R < 1/2$. Thus, R_x2 is indicated for large m.

R_x3. Let $m = H + R$, where $H \in \{0,1/2,1,3/2, \ldots\}$ and $0 < R < 1/2$. One follows the steps:

1. If $r \leq e/(e + r)$, go to (2). Otherwise go to (3).
2. Set $\xi = (r')^{1/R}$. Accept ξ if $r'' \leq e^{-\xi}$ and go to (4). Otherwise return to (1).
3. Set $\xi = 1 - \ell n r'$. Accept ξ if $\ell n r'' \leq (R - 1)\, \ell n \xi$ and go to (4). Otherwise return to (1).
4. If $H = 0$, set $x = \xi$ (accepted) and exit with x. If $H \geq 1/2$, sample $(\xi')^{H-1}e^{-\xi'}/\Gamma(H)$ for ξ' on $(0,\infty)$ by C45 or C64. Exit with $x = \xi + \xi'$. (Ahrens.)

J3. One can write $q(\xi) = \xi^{R-1}e^{-\xi}/\Gamma(R)$ as $q(\xi) = A^{-1}p_1(\xi)h(\xi)$, in the form of R13, where $A = e\Gamma(R + 1)/(e + R)$, and $p_1(\xi)$, $h(\xi)$ are both composite functions, namely

$$p_1(\xi) = \begin{cases} eR\xi^{R-1}/(e + R); & (0,1) \\ eRe^{-\xi}/(e + R); & (1,\infty), \end{cases}$$

a density function since $\int_0^\infty p_1(\xi)\, d\xi = \int_0^1 p_1(\xi)\, d\xi + \int_1^\infty p_1(\xi)\, d\xi$

$= ((eR)/e + R)(1/R + 1/e) = 1$, and

$$h(\xi) = \begin{cases} e^{-\xi}; & (0,1) \\ \xi^{R-1}; & (1,\infty), \end{cases}$$

where $0 < h(\xi) \leq 1$.

The rule samples $q(\xi)$ for ξ on $(0,\infty)$ by R13. The density $p_1(\xi)$ is sampled by C113, with $A_1 = \int_0^1 p_1(\xi)\, d\xi = e/(e + R)$, $A_2 = R/(e + R)$. In step (2), the setting of ξ uses C16. In (3), we use C1 to set r' $= \int_\xi^\infty e \cdot e^{-\xi}\, d\xi = e^{1-\xi}$, obtaining $\xi = 1 - \ln r'$. If $H = 0$, $R = m$, $q(\xi)$ $= \xi^{m-1} e^{-\xi}/\Gamma(m)$ and we exit with $x = \xi$ in step (4). If $H \geq 1/2$, we sample $(\xi')^{H-1} e^{-\xi'}/\Gamma(H)$ for ξ' and set $x = \xi + \xi'$ as explained in J2.

Note 3. The efficiency in R_x3 is

$A = e\Gamma(R + 1)/(e + R) \geq .74$.

R_x4. Let $m = H + R$ as in R_x3. Precompute $A_1 = \int_0^1 \xi^{R-1} e^{-\xi}\, d\xi/\Gamma(R)$, $A_2 = 1 - A_1$. (See [27].) One then follows the steps:

1. If $r \leq A_1$, go to (2). Otherwise go to (3).
2. Set $\xi = (r')^{1/R}$. Accept ξ if $r'' \leq e^{-\xi}$. Otherwise, iterate (2).
3. Set $\xi = 1 - \ln r'$. Accept ξ if $\ln r'' \leq (R - 1)\ln \xi$. Otherwise, iterate (3).
4. For accepted ξ from (2) or (3), proceed as in step (4) of R_x3. (Cashwell.)

J4. The only difference from R_x3 lies in the sampling of $q(\xi) = \xi^{R-1} e^{-\xi}/\Gamma(R)$, which we now write in the form

$$q(\xi) = \begin{cases} A_1[(A_1R\Gamma(R))^{-1}(R\xi^{R-1})(e^{-\xi})]; & (0,1) \\ A_2[(A_2e\Gamma(R))^{-1}(ee^{-\xi})(\xi^{R-1})]; & (1,\infty), \end{cases}$$

where we use C113 directly on $q(\xi)$, regarded as a composite function, and sample $a_1(\xi)/A_1$ with probability A_1, and $a_2(\xi)/A_2$ with probability A_2. The latter two densities are each of the form in R13, and the rule follows.

Note 4. The average of the number of trials for acceptance is $A_1/A_1R\Gamma(R) + A_2/A_2e\Gamma(R) = (e + R)/e\Gamma(R + 1) = 1/A$ where A is the efficiency of R_x3.

R28. $B(v) = v^{m-1}(1 - v)^{n-1}/B(m,n)$; $(0,1)$,

$b(z) = z^{m-1}/(1 + z)^{m+n}B(m,n)$; $(0,\infty)$,

$q(\theta) = 2 \sin^{2m-1}\theta \cos^{2n-1}\theta/B(m,n)$; $(0,\pi/2)$, m,n not both in the set $\{1/2,1,3/2,2, \ldots\}$.

R_x. Sample $x^{m-1}e^{-x}/\Gamma(m)$ and $y^{n-1}e^{-y}/\Gamma(n)$ for x,y on $(0,\infty)$ by C45, C64, or R27. Set $v = x/(x + y)$, $z = x/y$, $\theta = \arcsin v^{1/2}$.

J. The rule follows from C75 J.

R29. $t(z) = 2z^{2m-1}e^{-z^2}/\Gamma(m)$; $(0,\infty)$, $m > 0$, $m \notin \{1/2,1,3/2,2, \ldots\}$.

R_x. Sample $p(x) = x^{m-1}e^{-x}/\Gamma(m)$ for x on $(0,\infty)$ by R27. Set $z = x^{1/2}$.

J. For $z = x^{1/2}$ one has $t(z)\,dz = p(x)\,dx$. The rule follows from C2.

R30. $p(\alpha'/\alpha,\theta)$; polarized Compton scattering.

R_x. A method for sampling the Klein-Nishina cross section for polarized photons is given in [10,15]. This involves C161 and R16, 17, q.v. The fit now used for C161 (cf. [11,16,14]) is an improvement on that in [10], and is cited in the later expanded version [15].

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APPENDIX

(Some Tricks of the Trade)

Since the straightforward method (D1, C1) of sampling is seldom practical, it may be of some interest to collect here a few of the more ingenious devices employed above.

1. By far the most frequently exploited procedure is the change of variable (C2). This is by no means a triviality, especially in the case of several variables, where is is the basis for sampling many important densities, in particular the Γ and B densities (C75, R27, R28) with their host of special cases, and the many-variable normal density (C160).

2. If a given density can be recognized as that for the value of some function under a second density which can be sampled, the task is easy (D5, D7, C5, C7). Indeed, this leads, via the geometric result of C8, to the sampling of a whole hierarchy of basic densities (cf. C29, Note). A second consequence is the special case in which the sum, product, or quotient of two variables plays the role of the function referred to (C9). This also has some remarkable applications (C31, C32, C75, Note 1, C73).

3. A given density q(y) can sometimes be identified as the marginal y-density of a two-variable density f(x,y), for which the other marginal density p(x), and the x-dependent y-density $p(y|x)$, can both be sampled (D24, D33, C124). Sampling q(y) can then be effected. Numerous examples are given in the text. The variations in C130 and C135 are noteworthy. The latter leads to a remarkable method for sampling the "tail-end" density by way of its "first moment" density (C136).

4. A density which is a sum of positive terms may sometimes be sampled by sampling the densities defined by its normed terms (C3, C17, C35). The sampling of an interpolated density, a problem frequently occurring in practice, may be neatly accomplished by C3.

5. The device of C10 may always be used for sampling a linear density (C12) and sometimes for quadratic densities (C13), thus obviating the inversion of cubic distributions.

6. As a consequence of C12, it appears that the density $p(\nu) = 2\nu$ on (0,1) may be sampled by setting $\nu = \max\{r_1, r_2\}$ instead of the standard $\nu = r^{1/2}$. Generalization to the case $\nu = r^{1/n}$, with $\nu = \max\{r_1, \ldots, r_n\}$ is provided by C15, and by C144, which also reveals the significance of $\nu = \min\{r_1, \ldots, r_n\}$, in place of $\nu = 1 - (1 - r)^{1/N}$.

7. The method of sampling the unit N-sphere (C90) is indispensable for large N, and follows by a long chain of derivations from those in (2) above.

8. When all else fails, the rejection techniques may prove adequate. That most frequently used is given in R13, of which R20 is one of many consequences.

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