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by

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## ABSTRACT

It has become desirable to include, in the geometric subroutine of Monte Carlo programs, a procedure for finding the intersections of a line of flight with a toroidal surface. Such problems, for surfaces of third or fourth degree, depend in an obvious way on the solution of cubic or quartic equations. Although the latter subject is centuries old, we give here, without apology, a careful exposition of its details, required for machine computation, and not to be found elsewhere. A routine is then derived for solution of the required torus-intersection problem.

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1. The reduced cubic of a cubic. In the Taylor expansion of the complex cubic

$$f(x) = d + cx + bx^2 + x^3 = \sum_0^3 f^{(k)}(x_o)(x-x_o)^k/k!$$

one has $f''(x_o) = 2b + 6x_o = 0$ for $x_o = -b/3$, and then,

$$p = f'(x_o) = c - (b^2/3),$$

$$q = f(x_o) = d - (b/9)(c+2p). \tag{1}$$

Theorem 1. For $x = y - (b/3)$, and the p,q of (1) we have the identity

$$x^3 + bx^2 + cx + d = y^3 + py + q. \tag{2}$$

2. Roots of the reduced cubic. For the reduced cubic (2), we define $W = (p/3)^3 + (q/2)^2$, and for the sake of uniformity,

$$V = \begin{cases} W^{1/2} \text{ if } p \neq 0 \\ -q/2 \text{ if } p = 0. \end{cases}$$

Note: $a^{1/n}$ means the principal root of $z^n = a$, the principal root of any equation referring to its greatest real root, if any, otherwise to the non-real root of greatest magnitude, and least argument $\theta$ on $(0^\circ, 360^\circ)$.

We verify at once the general relations

$$V^2 = W, \text{ and } (-q/2+V)(-q/2-V) = (-p/3)^3. \tag{3}$$

Now if $p = 0$, (2) reads $y^3 + q = 0$, its roots being $H = (-q)^{1/3}$, $\omega H$, and $\omega^2 H$, where

$$\omega = (-1+i\sqrt{3})/2,\ \omega^2 = (-1-i\sqrt{3})/2, \text{ and } \omega^3 = 1.$$

Suppose $p \neq 0$ in the reduced cubic (2), let y be one of its roots, and z the principal root of the quadratic $z^2 - yz - (p/3) = 0$, so that $z \neq 0$, and

$$y = z - (p/3z). \tag{4}$$

Substitution of (4) in (2) shows that

$$z^6 + qz^3 - (p/3)^3 = \{z^3+(q/2)\}^2 - W = 0.$$

Hence z must satisfy

$$z^3 = -q/2 + V \quad \text{or} \quad z^3 = -q/2 - V.$$

Since $p \neq 0$ in (3), each factor has 3 distinct cube roots, $H_m$ and $J_m$, $m = 1,2,3$, z being one of these. Let

$$H = H_1 = (-q/2+V)^{1/3}. \tag{5}$$

Then the 3 distinct numbers $\zeta = HJ_m$ all satisfy $\zeta^3 = (-p/3)^3$ by (3), and some one must be $-p/3$ itself. We choose $J_1 = J$ where

$$J = (-p/3)/H \tag{6}$$

and list the $H_m$, $J_m$ in the order

$$H_m = H,\ \omega H,\ \omega^2 H; \quad J_m = J,\ \omega^2 J,\ \omega J \tag{7}$$

noting that all $H_m J_m = -p/3$. Since z must be one of the numbers (7), the original root y must be, by (4), one of the numbers

$$y_m = H_m + J_m; \quad m = 1,2,3. \tag{8}$$

The formulas (5), (6), (7), (8) actually yield the roots in general, as shown in

Theorem 2. A reduced cubic satisfies the identity

$$y^3 + py + q \equiv (y-y_1)(y-y_2)(y-y_3) \tag{9}$$

$$\text{where } y_1 = H + J,\ y_2 = \omega H + \omega^2 J,\ y_3 = \omega^2 H + \omega J$$
$$H = (-q/2+V)^{1/3}, \quad J = (-p/3)/H. \tag{10}$$

Proof. We need only compute the symmetric functions:

$$\Sigma y_1 = (1+\omega+\omega^2)(H+J) = 0$$

$$\Sigma y_1 y_2 = y_1(y_2+y_3) + (y_2 y_3)$$
$$= (\omega+\omega^2)(H+J)^2 + \{H^2+(\omega+\omega^2)HJ+J^2\}$$
$$= -(H+J)^2 + \{H^2-HJ+J^2\} = -3HJ = p$$
$$y_1(y_2 y_3) = (H+J)\{H^2-HJ+J^2\} = H^3 + J^3$$
$$= (-q/2+V) + (-q/2-V) = -q.$$

Note here that $J^3 = (-p/3)^3/H^3 = (-p/3)^3/(-q/2+V) = (-q/2-V)$ by (3).

3. The cubic discriminant. The discriminant $\Delta$ of a polynomial of degree n with roots $z_1,\cdots,z_n$ is defined as

$$\Delta = \prod_{1\le r<s\le n} (z_r - z_s)^2$$

and is invariant under a translation of the roots.

Theorem 3. The discriminant of the cubics (2) is

$$\Delta_3 = -4(27)W.$$

Proof. From Th. 2, we compute

$$y_1 - y_2 = (1-\omega)H - (1-\omega)\omega^2 J = (1-\omega)(H-\omega^2 J)$$
$$y_1 - y_3 = (1-\omega^2)H - (1-\omega^2)\omega J = (1-\omega^2)(H-\omega J)$$
$$y_2 - y_3 = (\omega-\omega^2)H - (\omega-\omega^2)J = (\omega-\omega^2)(H-J).$$

Since $(1-\omega)(1-\omega^2) = 3$, $\omega - \omega^2 = i\sqrt{3}$, and $(H-J)(H-\omega J)(H-\omega^2 J) = H^3 - J^3 = (-q/2+V) - (-q/2-V) = 2V$, we find $\Delta_3 = (3i\sqrt{3})^2(4V^2) = -4(27)W$. (Cf. (3).)

4. Nature of the roots of a real cubic. If all roots of a cubic are real and distinct, its discriminant $\Delta$ is obviously positive, and zero if any root is repeated. If one root r is real, and two form a conjugate nonreal pair $z,\bar{z}$, then

$$\Delta = \{(r-z)(r-\bar{z})(z-\bar{z})\}^2 = \{|r-z|^2 \cdot 2iI(z)\}^2 < 0.$$

Theorem 4. The nature of the roots of the real cubics (2) is indicated by the sign of their common discriminant thus:

I. $\Delta_3 > 0$ ($W < 0$) implies 3 real distinct roots.

II. $\Delta_3 = 0$ ($W = 0$) implies 3 real roots with duplication.

III. $\Delta_3 < 0$ ($W > 0$) implies one real root and a nonreal conjugate pair.

Proof. The cases at the right are the only possibilities for a real cubic.

5. Calculation of the cubic roots. For a real reduced cubic, the roots may be written more simply, under these cases:

Case I. $W < 0$. ($p < 0$ necessarily.) Here, $V = W^{1/2} = i\beta$, $\beta > 0$, and $-q/2 + V = -q/2 + i\beta = r(\cos\theta + i\sin\theta)$ for $r = \{(-q/2)^2+\beta^2\}^{\frac{1}{2}} = \{(-p/3)^{\frac{1}{2}}\}^3 > 0$ and $\theta = \text{arc}\cos(-q/2)/r$ on $(0^\circ,180^\circ)$. Hence $H = r^{\frac{1}{3}}(\cos\theta/3 + i\sin\theta/3)$, and $J = \bar{H}$ (since $H\bar{H} = -p/3$). The roots $y_m$ in Th. 2 are then $2R(H)$, $2R(\omega H)$, $2R(\omega^2 H)$, i.e.,

$$y_m = 2r^{1/3}\cos\psi_m$$

where $\psi_1 = \theta/3 \in (0^\circ,60^\circ)$, $\psi_2 = 120^\circ + \theta/3 \in (120^\circ, 180^\circ)$, and $\psi_3 = 240^\circ + \theta/3 \in (240^\circ,300^\circ)$. Since $\cos\psi_3 = \cos(360^\circ-\psi_3) = \cos(120^\circ-\theta/3)$, we may write the 3 real roots in the decreasing order

$$y = 2(-p/3)^{\frac{1}{2}}\cos\{\theta/3, 120^\circ-\theta/3, 120^\circ+\theta/3\}$$

with angles on the intervals $(0^\circ,60^\circ)$, $(60^\circ,120^\circ)$, $(120^\circ,180^\circ)$, respectively.

Case II. $W = 0$. ($p \leq 0$ necessarily.) In either case, we verify that $V = 0$, so $H = (-q/2)^{\frac{1}{3}}$, $J = H$ (since $H \cdot H = \{(q/2)^2\}^{\frac{1}{3}} = -p/3$). The roots are therefore $H + H = 2H$, and $\omega H + \omega^2 H = \omega^2 H + \omega H = -H$. Since $(q/2)^2 = (-p/3)^3 \geqq 0$, we prefer to compute $K = -H = (\text{sgn } q)(-p/3)^{\frac{1}{2}}$, and list the roots as $K, K, -2K$.

Case III. $W > 0$. Here, one verifies $q/2 \neq V \neq 0$ (for $p = 0$ or not), so

$$H = (-q/2+V)^{\frac{1}{3}} \neq 0 \quad \text{and} \quad J = (-p/3)/H \neq H$$

(since $J^3 = -q/2 - V \neq -q/2 + V = H^3$). The roots are therefore $H + J$ (real) and $\frac{1}{2}\{-(H+J) \pm i\sqrt{3}(H-J)\}$ (conjugate nonreal).

Summary for all roots of the real cubics (2): $x^3 + bx^2 + cx + d = y^3 + py + q$ where $x = y - b/3$ relates the roots, and $p = c - (b^2/3)$, $q = d - (b/9)(c+2p)$.

$$W = (p/3)^3 + (q/2)^2, \quad V = \begin{cases} W^{1/2} \text{ for } p \neq 0 \\ -q/2 \text{ for } p = 0. \end{cases}$$

I. $W < 0$ $(p < 0)$ $\theta = \arccos[(-q/2)/P^3] \in (0^\circ, 180^\circ)$,

$$P = (-p/3)^{\frac{1}{2}},$$

$y = 2P\cos\{\theta/3; 120^\circ \mp \theta/3\}$ real, distinct, decreasing.

II. $W = 0$ $(p \leqq 0)$ $y = K, K, -2K$, $K = (\text{sgn } q)P$,

$$P = (-p/3)^{\frac{1}{2}}.$$

| | | |
|---|---|---|
| Specifically, | $-P = -P < 0 < 2P$ | for $q < 0$ |
| | $-P = -P = 0 = 2P$ | for $q = 0$ |
| | $-2P < 0 < P = P$ | for $q > 0$. |

III. $W > 0$. $H = (-q/2+V)^{\frac{1}{3}} \neq 0$, $J = (-p/3)/H \neq H$

$$y = H + J, \ \tfrac{1}{2}\{-(H+J) \pm i\sqrt{3}(H-J)\}.$$

Note that case III is the only one involving a cube root, or reference to the definition of V.

6. The reduced quartic of a quartic. In the Taylor expansion of the complex quartic

$$F(x) = E + DX + CX^2 + BX^3 + X^4 = \Sigma_0^4 F^{(k)}(X_0)(X-X_0)^k/k!$$

one has $F^{(3)}(X_0) = 6B + 24X_0 = 0$ for $X_0 = -B/4$, and then

$$Q = F''(X_0)/2! = C - (3/2)(B/2)^2,$$

$$R = F'(X_0) = D + (B/2)\{(B/2)^2 - C\}$$

$$S = F(X_0) = E - (1/16)(B/2)\{5D - C(B/2) + 3R\}. \qquad (11)$$

Theorem 5. For $X = Y - (B/4)$, and the Q,R,S of (11), we have the identity

$$X^4 + BX^3 + CX^2 + DX + E = Y^4 + QY^2 + RY + S. \qquad (12)$$

7. Quadratic factorization of the reduced quartic. We assume $R \neq 0$ in (12) until §10, and seek numbers $k \neq 0, \ell, m$ such that

$$Y^4 + QY^2 + RY + S = (Y^2+kY+\ell)(Y^2-kY+m) \qquad (13)$$

shall be an identity in Y. For this we require $m + \ell = k^2 + Q$, $m - \ell = R/k$, $m\ell = S$ i.e., $2\ell = k^2 + Q - R/k$, $2m = k^2 + Q + R/k$, with product $k^4 + 2Qk^2 + Q^2 - R^2/k^2 = 4S$, so that $k^6 + 2Qk^4 + (Q^2-4S)k^2 - R^2 = 0$. The desired k must therefore satisfy $k^2 = x$, where x is a root (necessarily nonzero) of the cubic

$$f(x) = x^3 + 2Qx^2 + (Q^2-4S)x - R^2; \quad R \neq 0. \qquad (14)$$

Conversely, for any such $x_1, k_1$, and the corresponding $\ell_1, m_1$, (13) splits into the two quadratic factors

$$(Y+\tfrac{1}{2}k_1)^2 - \tfrac{1}{4}(T+U); \qquad (Y-\tfrac{1}{2}k_1)^2 - \tfrac{1}{4}(T-U)$$

with the roots

$$Y_n: \ \tfrac{1}{2}(-k_1 \pm (T+U)^{\frac{1}{2}}); \quad \tfrac{1}{2}(k_1 \pm (T-U)^{\frac{1}{2}}) \qquad (15)$$

where we have set

$$T = -(x_1+2Q); \quad U = 2R/k_1. \qquad (16)$$

8. The quartic discriminant. Since $x_1 + x_2 + x_3 = -2Q$ and $\left(x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}x_3^{\frac{1}{2}}\right)^2 = R^2 \neq 0$ for the roots $x_n$ of (14), in any fixed order, we may define $k_1 = x_1^{\frac{1}{2}}$, $k_2 = x_2^{\frac{1}{2}}$, and $k_3 = \pm x_3^{\frac{1}{2}}$ so that $k_1k_2k_3 = R$, thus obtaining the relations

$$T = k_2^2 + k_3^2; \quad U = 2k_2k_3$$

and hence $T + U = (k_2+k_3)^2$; $T - U = (k_2-k_3)^2$. (17)

Note that $(T+U)^{\frac{1}{2}} = \pm(k_2+k_3)$ and $(T-U)^{\frac{1}{2}} = \pm(k_2-k_3)$. The roots $Y_n$ in (15) may therefore be expressed in the form

$$Y_{1,2} = \tfrac{1}{2}\{-k_1 \pm (k_2+k_3)\}; \quad Y_{3,4} = \tfrac{1}{2}\{k_1 \pm (k_2-k_3)\} \quad (18)$$

(we read the upper sign for the 1st subscript; the signs here are not necessarily correlated with those in (15)).

An obvious computation now shows that $\pi_{t<u}(Y_t-Y_u)^2 = \pi_{r<s}\left(k_r^2-k_s^2\right)^2 = \pi_{r<s}(x_r-x_s)^2$, and we have

Theorem 6. The discriminant $\Delta_4$ of the quartics (12):

$X^4+BX^3+CX^2+DX+E = Y^4+QY^2+RY+S$, $R \neq 0$, $X = Y-(B/4)$,

is equal to $\Delta_3$, the discriminant of the cubics

$$f(x) = x^3 + 2Qx^2 + (Q^2-4S)x - R^2 = x^3 + bx^2 + cx + d = y^3 + py + q \quad (19)$$

where $x = y - b/3$, $p = c - (b^2/3)$, $q = d-(b/9)(c+2p)$, namely, $\Delta = \Delta_4 = \Delta_3 = -4(27)W$, with $W = (p/3)^3+(q/2)^2$. Moreover, in the notation defined, the roots $Y_n$ of (12) may be expressed in the equivalent forms (15) and (18).

9. Roots of the real quartic. The roots $x_m$ of (19) are here restricted by the condition $x_1x_2x_3 = R^2 > 0$, which implies at least one positive real root, and (−−+), (+++) as the only sign possibilities when all 3 are real. We now choose notation so that

1. $x_1$ is a largest positive real root, and $x_1 \geqq x_2 \geqq x_3$ when all are real ($W \leq 0$). If $W > 0$, we take $x_2$ with argument on $(0^\circ,180^\circ)$, and $x_3 = \bar{x}_2$.

2. In all cases, $k_1 = x_1^{\frac{1}{2}}$, $k_2 = x_2^{\frac{1}{2}}$, $k_3 = \pm x_3^{\frac{1}{2}}$ so that $k_1k_2k_3 = R$, as before. This insures the formulas (17), (18).

With these provisos, we give a complete determination of the roots $Y_n$ of (12), using the results of §5 without explicit reference.

I. $W < 0$ (3 real distinct $x_i$).

(A) If condition CI: $\{Q < 0$ and $Q^2 - 4S > 0\}$ holds, then $f(x)$ in (19) alternates in sign, and no root $x_m$ is negative. Hence we must have $x_1 > x_2 > x_3 > 0$

$$k_1 = x_1^{\frac{1}{2}} > k_2 = x_2^{\frac{1}{2}} > |k_3| > 0,$$

$$k_3 = (\operatorname{sgn} R)x_3^{\frac{1}{2}}.$$

There are 4 distinct real roots, namely

$$Y_{1,2} = \tfrac{1}{2}\{-k_1 \pm (k_2+k_3)\} = \tfrac{1}{2}\{-k_1 \pm (T+U)^{\frac{1}{2}}\}$$

$$Y_{3,4} = \tfrac{1}{2}\{k_1 \pm (k_2-k_3)\} = \tfrac{1}{2}\{k_1 \pm (T-U)^{\frac{1}{2}}\}.$$

The signs in the two forms are here correlated, and the first shows that $Y_3 > Y_4 > Y_1 > Y_2$, the same order obtaining for the corresponding $X_n$.

Computation: As in §5, we obtain $y_1 = 2P\cos\theta/3$, and $x_1 = y_1 - b/3 > 0$, and $k_1 = x_1^{\frac{1}{2}} > 0$, in any case. Now:

Method I. One may compute T,U from (16), with $T \pm U > 0$ (cf. (17)), and the $Y_n$ from the second form above.

Method II. One may compute as in §5

$y_2 = 2P\cos(120^\circ-\theta/3) = -P(\cos\theta/3 + \sqrt{3}\sin\theta/3) >$
$y_3 = 2P\cos(120^\circ+\theta/3) = -P(\cos\theta/3 - \sqrt{3}\sin\theta/3)$, and obtain $x_1 > x_2 = y_2 - b/3 > x_3 = y_3 - b/3 > 0$, $k_2 = x_2^{\frac{1}{2}}$, $k_3 = (\operatorname{sgn} R)x_3^{\frac{1}{2}}$ and the $Y_n$ from the first form above. (Method II involves two more cosines, or one more square root, $\sin\theta/3 = +(1-\cos^2\theta/3)^{\frac{1}{2}}$, than Method I.)

(B) If CI fails, the $x_m$ are not all positive, since this would imply $2Q = -\sum x_1 < 0$ and $Q^2 - 4S = \sum x_1x_2 > 0$. Hence, in this case, we must have $x_1 > 0 > x_2 > x_3$, $k_1 = x_1^{\frac{1}{2}}$, $k_2 = i|x_2|^{\frac{1}{2}}$, and $k_3 = -(\operatorname{sgn} R)i|x_3|^{\frac{1}{2}}$. We then see from (18) that $(Y_1,Y_2)$ and $(Y_3,Y_4)$ are different conjugate imaginary pairs, namely

$$Y_{1,2} = \tfrac{1}{2}\{-k_1 \pm i[|x_2|^{\frac{1}{2}}-(\operatorname{sgn} R)|x_3|^{\frac{1}{2}}]\}$$

$$Y_{3,4} = \tfrac{1}{2}\{k_1 \pm i[|x_2|^{\frac{1}{2}}+(\operatorname{sgn} R)|x_3|^{\frac{1}{2}}]\}.$$

If required, these may be obtained by computing $|x_2|^{\frac{1}{2}}$, $|x_3|^{\frac{1}{2}}$, as in Method II above, or from (15), (16); i.e.,

$$Y_n = \tfrac{1}{2}\{-k_1 \pm i[-(T+U)]^{\frac{1}{2}}\}$$

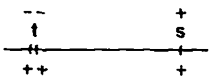
$$\tfrac{1}{2}\{k_1 \pm i[-(T-U)]^{\frac{1}{2}}\}$$

where $-(T\pm U) > 0$ (cf. (17)). Here the signs may not be correlated with those of $Y_{1,2}$, $Y_{3,4}$.

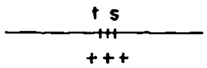
II. $W = 0$ (3 real $x_m$, with duplication).

The $x_m$ <u>sign</u> possibilities are $(--+)$ and $(+++)$. On the other hand, we know from §5 that the $x_m$ must be of the forms:

1. $t = -P - b/3 < s = 2P - b/3$ if $q < 0$




2. $t = -b/3 = s$ if $q = 0$




3. $s = -2P - b/3 < t = P - b/3$ if $q > 0$


where s and t denote roots of multiplicity 1 and 2 respectively, and $P = (-p/3)^{\frac{1}{2}} \geq 0$. It therefore appears that the sign alternative $(--+)$ occurs if and only if $q < 0$ <u>and</u> $t < 0$. Thus we have again two subcases; the simpler we treat <u>first</u> as (sic!)

(B). If condition CII: $\{q < 0$ <u>and</u> $t < 0\}$ holds, then $s = x_1 > 0 > x_2 = x_3 = t$, $k_1 = \sqrt{s}$, $k_2 = i\sqrt{|t|}$, $k_3 = -(\mathrm{sgn}\ R)k_2$. The roots (18) are then

(a) For $R > 0$, $Y_{1,2} = -\sqrt{s}/2$ doublet,

$Y_{3,4} = \sqrt{s}/2 \pm i\sqrt{|t|}$.

(b) For $R < 0$, $Y_{1,2} = -\sqrt{s}/2 \pm i\sqrt{|t|}$,

$Y_{3,4} = \sqrt{s}/2$ doublet.

Hence <u>two</u> $Y_n$ <u>are real and equal, and two are non-real conjugates</u>.

(A) If condition CII <u>fails</u>, then we must have the sign case $(+++)$, with $x_1 \geq x_2 \geq x_3 > 0$, and $k_1 = x_1^{\frac{1}{2}}$, $k_2 = x_2^{\frac{1}{2}}$, $k_3 = (\mathrm{sgn}\ R)x_3^{\frac{1}{2}}$, as in (IA).

<u>All</u> $Y_n$ <u>are real, but with duplication</u>, since $\Delta_4 = 0$. In detail, we have the following possibilities in (18):

(1) $q < 0$; $s = x_1 > x_2 = x_3 = t > 0$; $k_1 = \sqrt{s}$, $k_2 = \sqrt{t}$, $k_3 = (\mathrm{sgn}\ R)k_2$

(a) $R > 0$; $Y_{1,2} = -\sqrt{s}/2 \pm \sqrt{t}$,

$Y_{3,4} = \sqrt{s}/2$ doublet

(b) $R < 0$; $Y_{1,2} = -\sqrt{s}/2$ doublet,

$Y_{3,4} = \sqrt{s}/2 \pm \sqrt{t}$.

(2) $q = 0$; $s = x_1$ $(= x_2 = x_3 = t) > 0$;

$k_1 = \sqrt{s} = k_2 = \sqrt{t}$, $k_3 = (\mathrm{sgn}\ R)k_2$

(a) $R > 0$; $Y_{1,2} = \sqrt{s}/2,\ -3\sqrt{s}/2$, $Y_{3,4} = \sqrt{s}/2$

(b) $R < 0$; $Y_{1,2} = -\sqrt{s}/2$, $Y_{3,4} = 3\sqrt{s}/2,\ -\sqrt{s}/2$.

(3) $q > 0$; $t = x_1 = x_2 > x_3 = s > 0$;

$k_1 = \sqrt{t} = k_2$, $k_3 = (\mathrm{sgn}\ R)\sqrt{s}$

(a) $R > 0$; $Y_{1,2} = \sqrt{s}/2,\ -\sqrt{s}/2 - \sqrt{t}$,

$Y_{3,4} = -\sqrt{s}/2 + \sqrt{t}$, $\sqrt{s}/2$

(b) $R < 0$; $Y_{1,2} = -\sqrt{s}/2,\ \sqrt{s}/2 - \sqrt{t}$,

$Y_{3,4} = \sqrt{s}/2 + \sqrt{t}$, $-\sqrt{s}/2$.

Hence, if $q \lessgtr 0$, the $Y_n$ are

(a) for $R > 0$, $\sqrt{s}/2$ doublet, $-\sqrt{s}/2 \pm \sqrt{t}$

(b) for $R < 0$, $-\sqrt{s}/2$ doublet, $\sqrt{s}/2 \pm \sqrt{t}$.

If $q = 0$, then the $Y_n$ are

(a) for $R > 0$, $\sqrt{s}/2$ triplet, $-3\sqrt{s}/2$

(b) for $R < 0$, $-\sqrt{s}/2$ triplet, $3\sqrt{s}/2$.

III. $W > 0$ ($x_1 > 0$; $x_2, x_3 = \bar{x}_2$ nonreal conjugates). Under our provisos, we shall have $k_1 = x_1^{\frac{1}{2}} > 0$, $k_2 = x_2^{\frac{1}{2}} = \xi + i\eta$ with $\xi, \eta > 0$, and $k_3 = (\mathrm{sgn}\ R)\bar{k}_2 = (\mathrm{sgn}\ R)(\xi - \eta i)$. The roots (18) are then

(a) For $R > 0$; $Y_{1,2} = \tfrac{1}{2}\{-k_1 \pm 2\xi\}$,

$Y_{3,4} = \tfrac{1}{2}\{k_1 \pm 2i\eta\}$.

(b) For $R < 0$; $Y_{1,2} = \frac{1}{2}\{-k_1 \pm 2i\eta\}$,

$Y_{3,4} = \frac{1}{2}\{k_1 \pm 2\xi\}$.

Thus there are two distinct real roots, and a pair of nonreal conjugates.

Computation: As in §5, we find $H = (-q/2+V)^{\frac{1}{3}}$ real $\neq 0$, $J = (-p/3)/H$ real $\neq H$, $y_1 = H + J$, and $x_1 = y_1 - b/3 > 0$, $k_1 = \sqrt{x_1} > 0$. Now:

(a) For $R > 0$; $T + U = (k_2+k_3)^2 = (2\xi)^2 > 0$,

$\xi > 0$ implies $2\xi = (T+U)^{\frac{1}{2}} > 0$.

Similarly, $T - U = (k_2-k_3)^2 = (2i\eta)^2$

$= -(2\eta)^2$ implies $2\eta = [-(T-U)]^{\frac{1}{2}} > 0$.

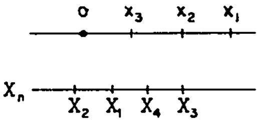
(b) For $R < 0$; one finds $2\eta = [-(T+U)]^{\frac{1}{2}} > 0$,

$2\xi = (T-U)^{\frac{1}{2}} > 0$ in the same fashion.

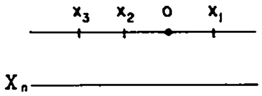
This gives the $2\xi, 2\eta$, required for the $Y_n$, in terms of $T \pm U$ computed from (16).

Schematics of real roots of the real quartic.

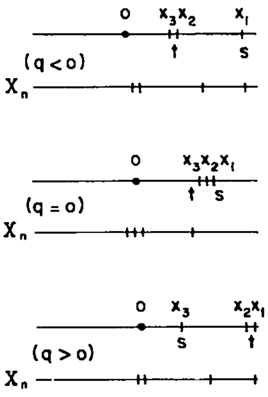
I. $W < 0$ (A)



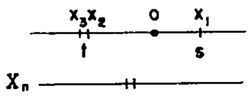

(B)



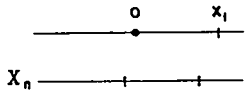

II. $W = 0$ (A)




(B)




III. $W > 0$




The $X_n$ are not ordered except in IA.

10. Procedure for real roots of the real quartic $X^4 + BX^3 + CX^2 + DX + E = 0$.

1. $Q = C - (3/2)(B/2)^2$, $R = D + (B/2)\{(B/2)^2 - C\}$,
   $S = E - (1/16)(B/2)\{5D - C(B/2) + 3R\}$.
2. $R \neq 0 \rightarrow (3)$ $\quad R = 0 \rightarrow (13)$.
3. $b = 2Q$, $c = Q^2 - 4S$, $d = -R^2$, $p = c - (b^2/3)$,
   $q = d - (b/9)(c+2p)$, $W = (p/3)^3 + (q/2)^2$.
4. $W \leq 0 \rightarrow (5)$ $\quad W > 0 \rightarrow (12)$.
5. $P = (-p/3)^{\frac{1}{2}}$ $\quad W < 0 \rightarrow (6)$ $\quad W = 0 \rightarrow (8)$.
6. $\{b < 0 \ \& \ c > 0\} \rightarrow (7)$
   $\{b < 0 \ \& \ c \leq 0\}$ or $\{b \geq 0\} \rightarrow$ No real $X_n$.
7. $\theta = \arccos(-q/2)/P^3 \in (0^\circ, 180^\circ)$
   $x = -b/3 + 2P\cos\theta/3$, $\quad k = \sqrt{x}$
   $T = -(x+b)$, $\quad U = 2R/k$
   $Y_{1,2} = \frac{1}{2}\{-k \pm \sqrt{T+U}\}$ $\quad Y_{3,4} = \frac{1}{2}\{k \pm \sqrt{T-U}\}$,
   $X_n = Y_n - (B/4)$.
   $(X_3 > X_4 > X_1 > X_2)$
8. $\begin{cases} q < 0 \rightarrow s = 2P - b/3, \ t = -P - b/3 \\ q \geq 0 \rightarrow s = -2P - b/3, \ t = P - b/3 \end{cases} k = \sqrt{s}/2 \rightarrow (9)$.
9. $\{q < 0 \ \& \ t < 0\} \rightarrow (10)$
   $\{q < 0 \ \& \ t \geq 0\}$ or $\{q \geq 0\} \rightarrow (11)$.
10. $\begin{cases} R > 0 \rightarrow X = -k - B/4) & \text{(doublet)} \\ R < 0 \rightarrow X = k - (B/4) & \text{(doublet).} \end{cases}$
11. $\{q \neq 0\} \rightarrow (11.1)$ $\quad \{q = 0\} \rightarrow (11.2)$

11.1 $k' = \sqrt{t} \rightarrow \begin{cases} R>0 \rightarrow Y = k, -k \pm k' \\ R<0 \rightarrow Y = -k, k \pm k' \end{cases} \rightarrow X = Y - B/4$ (1st doublet)

11.2 $\left.\begin{cases} R > 0 \rightarrow Y = k,\ -3k \\ R < 0 \rightarrow Y = -k,\ 3k \end{cases}\right\} \rightarrow X = Y - B/4$ (1st triplet).

12. $\left.\begin{cases} p \neq 0 \rightarrow V = W^{\frac{1}{2}} \\ p = 0 \rightarrow V = -q/2 \end{cases}\right\} \rightarrow H = (-q/2+V)^{\frac{1}{3}},$

$J = (-p/3)/H,\ y = H+J \rightarrow x = y-b/3,\ k = \sqrt{x}\ ,$

$T = -(x+b),\ U = 2R/k \rightarrow$

$\left.\begin{cases} R > 0 \rightarrow Y = \frac{1}{2}\{-k\pm\sqrt{T+U}\} \\ R < 0 \rightarrow Y = \frac{1}{2}\{k\pm\sqrt{T-U}\} \end{cases}\right\} \rightarrow X = Y - B/4$

(two real, distinct).

The following provide the real roots in the trivial case $R = 0$.

13. $b = Q/2,\ c = b^2 - S \rightarrow (14)$.

14. $c < 0$ (No real $X_i$) $c = 0 \rightarrow (15)$ $c > 0 \rightarrow (18)$.

15. $b > 0$ (No real $X_i$) $b = 0 \rightarrow (16)$ $b < 0 \rightarrow (17)$.

16. $X_i = -B/4$ (1 4-tuplet).

17. $X_i = \pm\sqrt{-b} - (B/4)$ (2 distinct doublets).

18. $d = \sqrt{c}\ ,\ r = -b + d \rightarrow (19)$.

19. $r < 0$ (No real $X_i$) $r = 0 \rightarrow (20)$ $r > 0 \rightarrow (21)$.

20. $X_i = -(B/4)$ (1 real doublet).

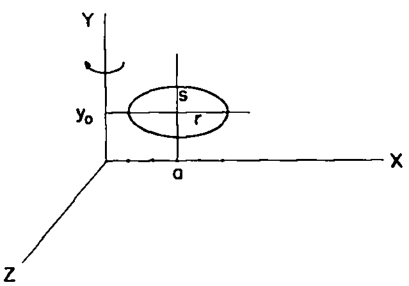
21. $s = -b - d$ $s < 0 \rightarrow (22)$ $s = 0 \rightarrow (23)$ $s > 0 \rightarrow (24)$.

22. $X_i = -(B/4) \pm \sqrt{r}$ (2 distinct singlets).

23. $X_i = -(B/4) + \{0, \pm\sqrt{r}\}$ (3 distinct, 1st a doublet).

24. $X_i = -(B/4) \pm \sqrt{r},\ -(B/4) \pm \sqrt{s}$ (4 distinct).

11. Equation of the elliptical torus. If $r,s,a > 0$, and $y_o \gtreqless 0$ are arbitrary, then $(x-a)^2/r^2 + (y-y_o)^2/s^2 = 1$ is the equation of an ellipse in the X,Y-plane, centered at $(a,y_o)$, with X,Y semi-axis lengths r and s. Rotation of the ellipse about the Y-axis generates an elliptical torus, which we call proper if $a > r$, and degenerate if $a \leq r$. The two have essentially different "primitive" equations:




$$\{(x^2+z^2)^{\frac{1}{2}}-a\}^2/r^2 + (y-y_o)^2/s^2 = 1;\quad a > r \quad \text{(a)}$$
$$\{(x^2+z^2)^{\frac{1}{2}}\mp a\}^2/r^2 + (y-y_o)^2/s^2 = 1;\quad a \leq r. \quad \text{(b)} \qquad (20)$$

In the degenerate case, the upper sign yields the equation of the outer surface, the lower, that of the inner. In the limiting case $a = 0$, these surfaces coincide, and (20b) is an ellipsoid of revolution.

Writing $\rho = r^2/s^2 > 0$, $\rho \gtreqless 1$, equations (20) may be written in the form

$$x^2+z^2+\rho y^2-2\rho y_o y+B_o = \begin{cases} 2a(x^2+z^2)^{\frac{1}{2}} & a > r \\ \pm 2a(x^2+z^2)^{\frac{1}{2}} & a \leq r \end{cases} \qquad (21)$$

where $B_o = a^2 - r^2 + \rho y_o^2$. It is notable that, in squaring both sides, an extraneous factor, with no real solution x,y,z, may be introduced in the first case only, so in either case, a point (x,y,z) is on the complete surface if and only if

$$\{x^2+z^2+\rho y^2-2\rho y_o y+B_o\}^2 = A_o(x^2+z^2);\quad a \gtreqless r \qquad (22)$$

where $A_o = 4a^2$.

12. Intersection of a line with the torus. A point (x,y,z) on the line $\{x = \xi+\alpha X,\ y = \eta+\beta X,\ z = \zeta+\gamma X; -\infty < X < \infty\}$, through the point $(\xi,\eta,\zeta)$, with direction $(\alpha,\beta,\gamma)$, $\alpha^2+\beta^2+\gamma^2 = 1$, lies on the torus (22) if and only if X satisfies the quartic equation

$$\{[(1-\beta^2)+\rho\beta^2]X^2 + [2(\alpha\xi+\gamma\zeta) + 2\rho\beta\eta - 2\rho\beta y_o]X + [\xi^2+\zeta^2+\rho\eta^2-2\rho\eta y_o+B_o]\}^2$$
$$= A_o[(1-\beta^2)X^2+2(\alpha\xi+\gamma\zeta)X+(\xi^2+\zeta^2)]. \qquad (23)$$

Setting $F = 1-\beta^2$, $G = F+\rho\beta^2$, $L = 2(\alpha\xi+\gamma\zeta)$,

$M = L+2\rho\beta(\eta-y_o)$, $T = \xi^2+\zeta^2$,

$U = T+\rho\eta(\eta-2y_o)+B_o$, (23) becomes

$(GX^2+MX+U)^2 = A_o(FX^2+LX+T)$. Since $G = (1-\beta^2)(1)+\beta^2(\rho)$

Examples for a debug. $B = 2$, $X_1 = Y_1 - 1/2$ always.

| C | D | E | Q | R | S | b | c | d | p | q | W | $x_m$ | Real $Y_n$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $-3/2$ | $-\frac{5}{2}-\sqrt{6}$ | $-\frac{19}{16}-\frac{\sqrt{6}}{2}$ | $-3$ | $-\sqrt{6}$ | $-1/2$ | $-6$ | $11$ | $-6$ | $-1$ | $0$ | $-\frac{1}{27}$ | $1,2,3$ | $\frac{-\sqrt{3}\pm(\sqrt{2}-1)}{2},\frac{\sqrt{3}\pm(\sqrt{2}+1)}{2}$ | IA |
| $-3$ | $-4+\sqrt{5}$ | $-\frac{5}{4}+\frac{\sqrt{5}}{2}$ | $-9/2$ | $\sqrt{5}$ | $-\frac{3}{16}$ | $-9$ | $21$ | $-5$ | $-6$ | $4$ | $-4$ | $2\pm\sqrt{3},5$ | $\frac{-\sqrt{5}\pm\sqrt{6}}{2},\frac{\sqrt{5}\pm\sqrt{2}}{2}$ | " |
| $3$ | $5$ | $\frac{13}{4}$ | $3/2$ | $3$ | $\frac{21}{16}$ | $3$ | $-3$ | $-9$ | $-6$ | $-4$ | $-4$ | $-3,\pm\sqrt{3}$ | None | IB |
| $2$ | $0$ | $0$ | $\frac{1}{2}$ | $-1$ | $\frac{5}{16}$ | $1$ | $-1$ | $-1$ | $-\frac{4}{3}$ | $-\frac{16}{27}$ | $0$ | $-1,-1,1$ | $\frac{1}{2},\frac{1}{2}$ | IIB |
| $2$ | $2$ | $1$ | $\frac{1}{2}$ | $1$ | $\frac{5}{16}$ | $1$ | $-1$ | $-1$ | $-\frac{4}{3}$ | $-\frac{16}{27}$ | $0$ | $-1,-1,1$ | $-\frac{1}{2},-\frac{1}{2}$ | " |
| $-\frac{1}{2}$ | $-\frac{3}{2}-\sqrt{2}$ | $-\frac{11}{16}-\frac{\sqrt{2}}{2}$ | $-2$ | $-\sqrt{2}$ | $-\frac{1}{4}$ | $-4$ | $5$ | $-2$ | $-\frac{1}{3}$ | $-\frac{2}{27}$ | $0$ | $1,1,2$ | $-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\pm 1$ | IIA |
| $-\frac{1}{2}$ | $-\frac{3}{2}+\sqrt{2}$ | $-\frac{11}{16}+\frac{\sqrt{2}}{2}$ | $-2$ | $\sqrt{2}$ | $-\frac{1}{4}$ | $-4$ | $5$ | $-2$ | $-\frac{1}{3}$ | $-\frac{2}{27}$ | $0$ | $1,1,2$ | $\frac{\sqrt{2}}{2},\ \frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\pm 1$ | " |
| $-1$ | $-4$ | $-2$ | $-\frac{5}{2}$ | $-2$ | $-\frac{7}{16}$ | $-5$ | $8$ | $-4$ | $-\frac{1}{3}$ | $\frac{2}{27}$ | $0$ | $1,2,2$ | $-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\pm\sqrt{2}$ | " |
| $-1$ | $0$ | $0$ | $-\frac{5}{2}$ | $2$ | $-\frac{7}{16}$ | $-5$ | $8$ | $-4$ | $-\frac{1}{3}$ | $\frac{2}{27}$ | $0$ | $1,2,2$ | $\frac{1}{2},\frac{1}{2},-\frac{1}{2}\pm\sqrt{2}$ | " |
| $0$ | $-2$ | $-1$ | $-\frac{3}{2}$ | $-1$ | $-\frac{3}{16}$ | $-3$ | $3$ | $-1$ | $0$ | $0$ | $0$ | $1,1,1$ | $-\frac{1}{2},-\frac{1}{2},\ -\frac{1}{2},\frac{3}{2}$ | " |
| $0$ | $0$ | $0$ | $-\frac{3}{2}$ | $1$ | $-\frac{3}{16}$ | $-3$ | $3$ | $-1$ | $0$ | $0$ | $0$ | $1,1,1$ | $\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{3}{2}$ | " |
| $-3$ | $-4-\sqrt{19}$ | $-\frac{11}{4}-\frac{\sqrt{19}}{2}$ | $-\frac{9}{2}$ | $-\sqrt{19}$ | $-\frac{27}{16}$ | $-9$ | $27$ | $-19$ | $0$ | $8$ | $16$ | $1,4\pm i\sqrt{3}$ | $\frac{1}{2}\pm\left(2+\frac{\sqrt{19}}{2}\right)^{\frac{1}{2}}$ | III |
| $-3$ | $-4+\sqrt{19}$ | $-\frac{11}{4}+\frac{\sqrt{19}}{2}$ | $-\frac{9}{2}$ | $\sqrt{19}$ | $-\frac{27}{16}$ | $-9$ | $27$ | $-19$ | $0$ | $8$ | $16$ | $1,4\pm i\sqrt{3}$ | $-\frac{1}{2}\pm\left(2+\frac{\sqrt{19}}{2}\right)^{\frac{1}{2}}$ | " |
| $3$ | $2-\sqrt{7}$ | $\frac{1}{4}-\frac{\sqrt{7}}{2}$ | $\frac{3}{2}$ | $-\sqrt{7}$ | $-\frac{3}{16}$ | $3$ | $3$ | $-7$ | $0$ | $-8$ | $16$ | $1,-2\pm i\sqrt{3}$ | $\frac{1}{2}\pm\left(-1+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$ | " |
| $3$ | $2+\sqrt{7}$ | $\frac{1}{4}+\frac{\sqrt{7}}{2}$ | $\frac{3}{2}$ | $\sqrt{7}$ | $-\frac{3}{16}$ | $3$ | $3$ | $-7$ | $0$ | $-8$ | $16$ | $1,-2\pm i\sqrt{3}$ | $-\frac{1}{2}\pm\left(-1+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$ | " |
| $0$ | $-1-\sqrt{14}$ | $-2-\frac{\sqrt{14}}{2}$ | $-\frac{3}{2}$ | $-\sqrt{14}$ | $-\frac{27}{16}$ | $-3$ | $9$ | $-14$ | $6$ | $-7$ | $\frac{81}{4}$ | $2,\frac{1\pm 3i\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}\pm\left(\frac{1}{4}+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$ | " |
| $0$ | $-1+\sqrt{14}$ | $-2+\frac{\sqrt{14}}{2}$ | $-\frac{3}{2}$ | $\sqrt{14}$ | $-\frac{27}{16}$ | $-3$ | $9$ | $-14$ | $6$ | $-7$ | $\frac{81}{4}$ | $2,\frac{1\pm 3i\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}\pm\left(\frac{1}{4}+\frac{\sqrt{7}}{2}\right)^{\frac{1}{2}}$ | " |
| $-\frac{5}{2}$ | $-\frac{7}{2}$ | $\frac{65}{16}$ | $-4$ | $0$ | $5$ | $-2$ | $-1$ | | | | | | None | R=0 |
| $\frac{15}{2}$ | $\frac{13}{2}$ | $\frac{169}{16}$ | $6$ | $0$ | $9$ | $3$ | $0$ | | | | | | None | " |
| $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{1}{16}$ | $0$ | $0$ | $0$ | $0$ | $0$ | | | | | | $0,0,0,0$ | " |
| $-\frac{9}{2}$ | $-\frac{11}{2}$ | $\frac{121}{16}$ | $-6$ | $0$ | $9$ | $-3$ | $0$ | | | | | | $\sqrt{3},\sqrt{3},-\sqrt{3},-\sqrt{3}$ | " |
| $\frac{15}{2}$ | $\frac{13}{2}$ | $\frac{105}{16}$ | $6$ | $0$ | $5$ | $3$ | $4$ | | | | | | None | " |
| $\frac{11}{2}$ | $\frac{9}{2}$ | $\frac{17}{16}$ | $4$ | $0$ | $0$ | $2$ | $4$ | | | | | | $0,0$ | " |
| $-1/2$ | $-\frac{3}{2}$ | $-\frac{55}{16}$ | $-2$ | $0$ | $-3$ | $-1$ | $4$ | | | | | | $\pm\sqrt{3}$ | " |
| $-5/2$ | $-\frac{7}{2}$ | $-\frac{15}{16}$ | $-4$ | $0$ | $0$ | $-2$ | $4$ | | | | | | $0,0,\pm 2$ | " |
| $-9/2$ | $-\frac{11}{2}$ | $\frac{57}{16}$ | $-6$ | $0$ | $5$ | $-3$ | $4$ | | | | | | $\pm 1,\ \pm\sqrt{5}$ | " |

is "barycentric", with $0 \le \beta^2 \le 1$, G is between 1 and $\rho > 0$, hence $G > 0$. Defining $M' = M/G$, $U' = U/G$, $A = A_0/G^2$, the latter quartic becomes

$$(X^2+M'X+U')^2 = A(FX^2+LX+T),\text{ or}$$

$$X^4+2M'X^3+(M'^2+2U'-AF)X^2+(2M'U'-AL)X+(U'^2-AT) = 0. \quad (24)$$

<u>Theorem 7</u>. All points (x,y,z) of intersection of the ray $\{x = \xi+\alpha X,\ y = \eta+\beta X,\ z = \zeta+\gamma X;\ X > 0\}$ with the torus (22) are determined by the positive real roots X of the quartic

$$X^4 + BX^3 + CX^2 + DX + E = 0$$

where we set

$F = 1-\beta^2$, $L = 2(\alpha\xi+\gamma\zeta)$, $T = \xi^2+\zeta^2$,

$G = F+\rho\beta^2$, $A = A_0/G^2$, $M' = \{L+2\rho\beta(\eta-y_0)\}/G$,

$U' = \{T+\rho\eta(\eta-2y_0)+B_0\}/G$, and

$B = 2M'$, $C = M'^2+2U'-AF$, $D = 2M'U'-AL$, $E = U'^2-AT$.

Here $A_0 = 4a^2$, $B_0 = a^2-r^2+\rho y_0^2$ are stored constants of the torus.

Finally, we state without proof the obvious

<u>Theorem 8</u>. (a) An arbitrary point (x,y,z) is (properly) inside the outer surface of a torus, if and only if

$$x^2 + z^2 + \rho y^2 - 2\rho y_0 y + B_0 < 2a(x^2+z^2)^{\frac{1}{2}}.$$

(b) A point (x,y,z), on a degenerate torus $(a \le r)$ is on the (open) inner surface if and only if

$$x^2 + z^2 + \rho y^2 - 2\rho y_0 y + B_0 < 0.$$

Thus the points (x,y,z) of intersection of a ray with a degenerate torus may be tested for the part of the surface on which they lie.

## General Reference

L. E. Dickson, Elementary theory of equations (1914), John Wiley and Sons, Inc., New York, N.Y..